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BESSEL FUNCTIONS FOR ENGINEERS

BY

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D.Sc. (ENGINEERING) LONDON

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NOTE

UNFORTUNATELY war conditions have prevented the text from being revised and brought up to date. References to recent papers have been added on p. xii. New formulae are given on p. xi. In 1936 the author and A. L. Meyers introduced the ster and stei functions for evaluating integrals (5)–(8) on p. xi. They have been christened (irreverently perhaps!) the 'farmyard functions', a nomenclature which may aid memorization. The preparation of tables of polar values, and extension of those of $N_0(z)$, $\phi_0(z)$, $N_1(z)$, $\phi_1(z)$ in ref. 13, p. xii, would be useful work for a post-graduate student. In Chap. II the term 'period' signifies that the function is oscillatory, its larger zeros being almost equally spaced. Strictly a periodic function repeats itself *exactly* at regular intervals, e.g. $\sin z$, $\cos z$. Throughout the text certain infinite series have been differentiated and integrated term by term, since the appropriate conditions given in Bromwich's *Theory of Infinite Series* (1926) are satisfied. The conditions for validity of differentiation under the integral sign, changing the order of integration, and the derivation and theory of asymptotic series are also treated therein. The omission of contour integral representation of Bessel functions and its technical applications has been rectified by publication in 1939 of *Complex Variable and Operational Calculus with Technical Applications*. Asymptotic series and their derivation are treated therein. The definition of an asymptotic series on p. 300 should read $|z^{n-1}R_n| \to 0$ as $|z| \to \infty$, not $|R_n| \to 0$ as $|z| \to \infty$.

N. W. M.

LONDON, 1941

PREFACE

THE use of Bessel functions in research and design work is now so extensive that the theory and its applications may well find a place in the technical training of those destined to be engineers. The three well-known treatises by Gray, Mathews, and MacRobert [77], Neilsen [85], and Watson [93],† whilst of great value to pure and applied mathematicians, are not suitable for engineers, whose main

† The numbers in square brackets denote the references at the end of the book.

purpose is the application of the functions to practical problems. Consequently this volume has been specially written for engineers, so that they can become familiar with that part of the theory which is required in applied analysis. The book will be useful in connexion with Chapters II to VI of the author's treatise on *Loud Speakers*. At the same time it may also serve to introduce the functions to students of pure and applied mathematics. The treatment is simple yet rigorous enough for engineers, whilst the text contains many worked examples illustrating various analytical processes. No prior knowledge, beyond that which should be obtainable in an engineering degree course is required, and the sequence of the chapters has been arranged with this in view. The functions have not been introduced via Laplace's equation, since this approach is more in keeping with the outlook of the mathematician than with that of the engineer. For the benefit of teachers it may be said that the subject-matter has been used for a course of lectures to practising engineers with success. Owing to space limitations the theory and the more detailed parts of the practical applications have been curtailed in places, but nothing of fundamental importance to engineers has been omitted. For the same reason there is no mention of contour integrals [82] or of Heaviside's operators [79 a]. The reader who desires to supplement his knowledge can do so by aid of the reference list at the end of the book. Where necessary, practical analysis has been shorn of its technicalities, whilst references to the original works are given to enable the reader to complete his studies. Including subdivisions, there are approximately 600 examples to be worked out by the reader. Many of the examples are either practical problems or represent practical analysis dissected to effect simplification. Those devoid of a practical basis are included to facilitate understanding and memorization of the more important formulae. This is a *sine qua non* in mathematical work, since it is rather hazardous to solve practical problems with a book in one hand and a pen in the other without a proper knowledge of the processes involved. A list of important formulae is given for reference purposes on pp. 157–72, but additional formulae will be encountered in the examples. No attempt has been made to print a comprehensive set of tables, since these are available elsewhere.† The tables have been selected chiefly to suit the requirements of acoustical and electrical engineers, the entries being given to four significant figures, which is ample for engineering purposes.

† See list of reference books on p. 187.

The author is indebted to the Syndics of the Cambridge University Press for permission to use entries from G. N. Watson's *Bessel Functions* to complete Tables 3–7; to the Committee of the British Association for kind permission to publish Tables 8–11; also to Professor H. B. Dwight and the American Institute of Electrical Engineers for kind permission to reproduce Tables 12, 13 [37]. He is also much indebted to Mr. R. R. M. Mallock for Tables 14, 15, giving the polar values of the ber and bei functions, which are abridgements of five-figure tables computed by him in 1928. The polar form has been suggested by Kennelly, Laws, and Pierce [40], who computed tables which are reproduced in the Report of the British Association, 1923. Tables are also given in *Funktionentafeln* by Jahnke and Emde [79]. The polar form has been generalized in Chapter VIII and new formulae are developed which simplify analysis involving ber, bei, ker, and kei functions. The polar values of the ker and kei functions, viz. $N_0(z)$, $\phi_0(z)$, $N_1(z)$, $\phi_1(z)$, were not available when the manuscript was completed. They will be found in a paper by Mr. A. L. Meyers and the author, entitled 'The polar form of the ker and kei functions and its application to eddy current heating', which is published in the *Philosophical Magazine*, **18**, 610, 1934. Formulae given therein for calculating polar and cartesian values, for large and small arguments, have been reproduced on p. x, in examples 59–66, p. 133, and on pp. 179, 180.

Professor T. M. MacRobert has very kindly criticized the manuscript and read the proofs from the view-point of the pure mathematician, whilst Messrs. C. R. Cosens, R. R. M. Mallock, and A. L. Meyers have done likewise from the view-point of the engineer-mathematician. Mr. Meyers generously undertook the herculean task of checking the examples. The author has very great pleasure in expressing his appreciation of the excellent suggestions made by these gentlemen.

N. W. M.

LONDON,
September 1934

CONTENTS

SYMBOLS ix

PLANE WAVE ASSUMPTION IN HORN THEORY . . . x

POLAR FORMULAE FOR THE KER AND KEI FUNCTIONS . x

ADDITIONAL FORMULAE xi

ADDITIONAL REFERENCES xii

I. HISTORICAL INTRODUCTION; FUNCTIONS OF ORDER ZERO; VIBRATION OF MEMBRANES 1

II. FUNCTIONS OF HIGHER ORDERS 20

III. EXPANSIONS IN TERMS OF BESSEL FUNCTIONS: INTEGRATION 41

IV. THE HYPERGEOMETRIC, GAMMA, AND STRUVE FUNCTIONS; ASYMPTOTIC EXPANSIONS; LOUD-SPEAKER HORNS 56

V. ADDITIONAL INTEGRALS INVOLVING BESSEL FUNCTIONS 89

VI. LOMMEL INTEGRALS FOR PRODUCTS OF TWO BESSEL FUNCTIONS 94

VII. THE MODIFIED BESSEL FUNCTIONS $I_\nu(z)$ AND $K_\nu(z)$. 102

VIII. BER, BEI, KER, AND KEI FUNCTIONS 119

IX. APPLICATION OF BER AND BEI FUNCTIONS TO THE RESISTANCE OF CONDUCTORS TO ALTERNATING CURRENT 134

LIST OF FORMULAE 157

TABLES 173

REFERENCES 184

INDEX 188

SYMBOLS

THE symbols used for various physical quantities are defined in the text and are identical with those in *Loud Speakers*, although there are a few additions. Arbitrary constants are usually denoted by the notation A_1, B_1, C_1,... so that in general they will not be mistaken for A area, B flux density, C capacity,.... Following E. C. J. Lommel the order of a Bessel function is represented by m or n when integral, but by μ or ν when unrestricted. The symbols for the various Bessel functions are given below:

$J_n(z)$, $J_\nu(z)$	Bessel function of the first kind.
$Y_n(z)$, $Y_\nu(z)$	,, ,, ,, second kind, as defined by Weber.
$\Upsilon_n(z)$, $\Upsilon_\nu(z)$	,, ,, ,, ,, ,, ,, ,, Neumann.
$H_n^{(1)}(z)$, $H_\nu^{(1)}(z)$	,, ,, ,, third ,, ,, ,, Nielsen.
$H_n^{(2)}(z)$, $H_\nu^{(2)}(z)$	,, ,, ,, ,, ,, ,, ,, ,,
$\mathfrak{C}_n(z)$, $\mathfrak{C}_\nu(z)$	Cylinder function as defined by Sonine.
$I_n(z)$, $I_\nu(z)$	Modified Bessel function of the first kind.
$K_n(z)$, $K_\nu(z)$	,, ,, ,, ,, second kind.
$\mathbf{H}_n(z)$, $\mathbf{H}_\nu(z)$	Struve's function. $R(\nu)$ signifies 'the real part of' ν.
$M_n(z)$, $M_\nu(z)$	$\sqrt{(\mathrm{ber}_n^2 z+\mathrm{bei}_n^2 z)}$, $\sqrt{(\mathrm{ber}_\nu^2 z+\mathrm{bei}_\nu^2 z)}$.
$\theta_n(z)$, $\theta_\nu(z)$	$\tan^{-1}\dfrac{\mathrm{bei}_n z}{\mathrm{ber}_n z}$, $\tan^{-1}\dfrac{\mathrm{bei}_\nu z}{\mathrm{ber}_\nu z}$.
$N_n(z)$, $N_\nu(z)$	$\sqrt{(\mathrm{ker}_n^2 z+\mathrm{kei}_n^2 z)}$, $\sqrt{(\mathrm{ker}_\nu^2 z+\mathrm{kei}_\nu^2 z)}$.
$\phi_n(z)$, $\phi_\nu(z)$	$\tan^{-1}\dfrac{\mathrm{kei}_n z}{\mathrm{ker}_n z}$, $\tan^{-1}\dfrac{\mathrm{kei}_\nu z}{\mathrm{ker}_\nu z}$.

$\mathbf{W}(z) = \dfrac{2}{z}\dfrac{M_1(z)}{M_0(z)}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)$, the dissipation or loss function (Chap. IX).

$\mathbf{\Pi}(z) = \dfrac{2}{z}\dfrac{M_1(z)}{M_0(z)}\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi)$, the penetration function (Chap. IX).

In dealing with the ber and bei functions (Chap. VIII) the symbols $i^{\pm\frac{1}{2}}$, $i^{\pm\frac{3}{2}}$ are used to represent $e^{\pm\frac{1}{4}\pi i}$, $e^{\pm\frac{3}{4}\pi i}$, thereby avoiding the symbols $\sqrt{(i)}, \sqrt{(-i)}$ as these are apt to lead to confusion owing to the ambiguity in sign. Thus we take as standard functions $J_0(zi^{\frac{3}{2}}) = \mathrm{ber}\, z+i\,\mathrm{bei}\, z$ and $J_1(zi^{\frac{3}{2}}) = \mathrm{ber}_1 z+i\,\mathrm{bei}_1 z$, whereas some continental writers use $J_0(z\sqrt{i}) = J_0(zi^{\frac{1}{2}}) = \mathrm{ber}\, z-i\,\mathrm{bei}\, z$ and $J_1(z\sqrt{i}) = J_1(zi^{\frac{1}{2}}) = -\mathrm{ber}_1 z+i\,\mathrm{bei}_1 z$. When using tables, care must be taken to ascertain which functions have been tabulated. The values of the various complex quantities involved are shown in Fig. 18.

In works on pure mathematics some writers use amp z, whilst others prefer arg z to signify the angle of inclination of the vector representing a complex quantity, to the positive real axis. From the view-point of an engineer or a physicist, neither of these symbols says what it means, nor does it mean what it says! Amp signifies ampere or amplitude, and if the latter, the maximum value of an oscillation is implied, which corresponds to the modulus of the complex quantity, but not to an angle. Arg z implies the independent variable

or argument of a function such as z in $J_n(z)$. Consequently, to avoid ambiguity, we shall use 'phase z' or θ to denote the angle of the vector with the positive real axis. Thus if $z = x+iy$, θ = 'phase z' $= \tan^{-1}(y/x)$.

In problems relating to vibration or alternating currents, the time factor $e^{i\omega t}$ is tacitly implied, but for brevity it is usually omitted. It is introduced (in mind) prior to differentiation with respect to t, but it is removed immediately afterwards.

Symbols of the form $\int^{z}$ signify indefinite integrals.

PLANE WAVE ASSUMPTION IN HORN THEORY, § 13, CHAP. IV

The horn analysis is based on the assumption that the sound pressure is equal in amplitude and in phase over plane sections orthogonal to the linear axis. It is only tenable for a distance from the throat which decreases with (1) rise in frequency, (2) increase in throat area, (3) increase in the rate of expansion. The latter in a very long flared horn is ultimately large enough for the section to be regarded as a flat surface in an infinite plane, so the radiation is then projected in the form of a beam [83, Chap. V]. Within the horn itself reflection is neglected and the transmitted wave only is considered. This is permissible provided the rate of expansion and the frequency are not too high. In a practical horn, under this condition, any modification in the formula for throat impedance can usually be neglected, except that at low frequencies to embody the influence of reflection at the mouth. If we imagine the horn to be divided up into a myriad of frictionless conduits, of small dimensions compared with the wave-length, the analysis in Chap. IV is rigorous, provided the area refers to that curved surface all points of which, measured along the centre lines of the conduits, are equidistant from the throat.

POLAR FORMULAE FOR THE KER AND KEI FUNCTIONS

When $0 \leqslant z < 0{\cdot}2$,

$$N_0(z) \doteqdot B_0 - \frac{\pi}{4B_0}v^2 + \left\{\frac{A_0^2+\frac{5}{2}A_0+2+\frac{\pi^2}{16}\left(1-\frac{2}{B_0^2}\right)}{4B_0}\right\}v^4; \tag{1}$$

$$\phi_0(z) \doteqdot -\tan^{-1}\frac{\pi}{4A_0} + \left(1+\frac{A_0}{B_0^2}\right)v^2 + \frac{\pi}{4B_0^2}\left(\frac{5}{8}+\frac{A_0}{B_0^2}\right)v^4 \text{ radians}; \tag{2}$$

$$N_1(z) \doteqdot \frac{1}{2v} - \tfrac{1}{4}\pi v + (A_0^2+\tfrac{3}{2}A_0+\tfrac{7}{8})v^3; \tag{3}$$

$$\phi_1(z) \doteqdot \tfrac{5}{4}\pi - 2(A_0+\tfrac{1}{2})v^2 - \pi(A_0+\tfrac{3}{4})v^4 \text{ radians}; \tag{4}$$

where $\quad v = \tfrac{1}{2}z,\ A_0 = (0{\cdot}1159315\ldots - \log_e z),\ B_0^2 = A_0^2 + \tfrac{1}{16}\pi^2.$

ADDITIONAL FORMULAE

From additional reference 15, we have

$$\int^{z} (lz)^{\nu}J_{\nu}(lz)\,dz$$
$$= 2^{\nu-1}\sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2})z[J_{\nu+1}(lz)\mathbf{H}_{\nu}(lz)-\mathbf{H}_{\nu+1}(lz)J_{\nu}(lz)]+(lz)^{\nu+1}J_{\nu}(lz)/l(2\nu+1).$$

Replacing the J's by Y's gives the value of $\int^{z} (lz)^{\nu}Y_{\nu}(lz)\,dz$. By writing $\nu = 0$, $l = 1$, and taking the lower limit zero, formulae (42), (43) on p. 49 are obtained.

The following integrals are evaluated in the same paper: (1) $\int^{z} z^{\mu}J_{\nu}(z)\,dz$; (2) $\int^{z} (lz)^{\nu}I_{\nu}(lz)\,dz$; (3) $\int^{z} (lz)^{\nu}K_{\nu}(lz)\,dz$; (4) $\int^{z} e^{iz}z^{\nu}J_{\nu}(z)\,dz$; (5) $\int^{z} z^{\nu}\,\mathrm{ber}_{\nu}z\,dz$; (6) $\int^{z} z^{\nu}\,\mathrm{bei}_{\nu}z\,dz$; (7) $\int^{z} z^{\nu}\,\mathrm{ker}_{\nu}z\,dz$; (8) $\int^{z} z^{\nu}\,\mathrm{kei}_{\nu}z\,dz$. The first and second derivatives of the functions $M_{\nu}(z)$, $\theta_{\nu}(z)$, $N_{\nu}(z)$, $\phi_{\nu}(z)$ in Chap. VIII are given also. Integral (1) is expressed in terms of the Lommel dual order function $S_{\mu,\nu}(lz)$, treated in reference 93. Integrals (2), (3) involve the modified Struve function $\mathbf{L}_{\nu}(z) = i^{-\nu-1}\mathbf{H}_{\nu}(zi)$, whilst (5)–(8) are given in terms of ster and stei functions [see additional reference 14] defined thus:

$$\mathbf{H}_{\nu}(zi^{\frac{3}{2}}) = i^{\nu+1}\mathbf{L}_{\nu}(zi^{\frac{1}{2}}) = \mathrm{ster}_{\nu}z+i\,\mathrm{stei}_{\nu}z,$$
$$= S_{\nu}(z)e^{i\psi_{\nu}(z)} = S_{\nu}(z)[\cos\psi_{\nu}(z)+i\sin\psi_{\nu}(z)],$$

where $\quad S_{\nu}(z) = \sqrt{(\mathrm{ster}_{\nu}^{2}z+\mathrm{stei}_{\nu}^{2}z)}, \qquad \psi_{\nu}(z) = \tan^{-1}(\mathrm{stei}_{\nu}z/\mathrm{ster}_{\nu}z).$

Infinite integral. By § 11, Chap. IV, when z is real, positive, and large enough,

$$J_{\nu}(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\cos(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)-\left(\frac{4\nu^{2}-1}{8z}\right)\sin(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)\right\}.$$

Using partial integration and neglecting unwanted terms, we get

$$-\int_{z}^{\infty} J_{\nu}(z)\,dz \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\left(\frac{4\nu^{2}-5}{8z}\right)\cos(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)+\sin(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)\right\}.$$

Since $\int_{0}^{\infty} J_{\nu}(z)\,dz = 1$, $R(\nu) > -1$, when z is real, positive, and large enough,

$$\int_{0}^{z} J_{\nu}(z)\,dz \doteqdot 1+\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\left(\frac{4\nu^{2}-5}{8z}\right)\cos(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)+\sin(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)\right\}.$$

A closer approximation can be obtained by using more terms of the asymptotic formula prior to integration.

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HISTORICAL INTRODUCTION; FUNCTIONS OF ORDER ZERO; VIBRATION OF MEMBRANES

1. Introduction

BESSEL functions, like many other branches of mathematics, had their origin in the solution of purely physical problems. Particular cases of these functions occurred in the solutions of differential equations over a century before the advent of the famous memoir written by the German astronomer, F. W. Bessel in 1824 and published in 1826. Consequently the nomenclature 'Bessel Function' did not exist prior to 1826.

In 1732 Daniel Bernoulli, a Swiss mathematician, well known as one of three illustrious brothers, studied the problem of the oscillations of a heavy flexible chain suspended vertically with its lower end free. This end is assumed to be moved sideways slightly and then released, so Bernoulli's problem was to determine the subsequent motion of the chain. In doing this he obtained a differential equation of the form

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\frac{k^2y}{z}=0, \tag{1}$$

which is a particular case of an equation formulated by Bessel about a century later. Its solution is given in example 15 at the end of the present chapter.

L. Euler, so well known to engineers for his long strut formula, investigated the vibrations of a stretched circular membrane in 1764, and he obtained a differential equation identical in form with that now accepted as the generalized Bessel equation. During the solution of an astronomical problem in 1770, the French mathematician, J. L. de Lagrange arrived at an equation whose solution, presented in the form of an infinite series, involved coefficients now coupled with Bessel's name. An investigation of the coefficients obtained by Lagrange was conducted some years later by the Italian mathematician, F. Carlini, and the famous French analyst, P. S. de Laplace.

The year 1822 is of singular importance in the history of mathematics owing to the publication of J. B. Fourier's treatise on *The Analytical Theory of Heat*. This epoch-making analysis had adorned the archives of the Paris Academy of Sciences for about twelve

years. Its publication was delayed for fear that it should adversely affect the prestige of the powers that were. In treating a problem on the distribution of temperature in a cylinder, which was heated and then allowed to cool under certain conditions, Fourier obtained a particular case of a Bessel equation and gave its solution (zero order). Further analytical researches on the distribution of temperature in spheres and cylinders were published by the French mathematician, S. D. Poisson, so well known to engineers for his ratio $\sigma = \dfrac{\text{lateral strain}}{\text{longitudinal strain}}$. Poisson's work was associated with functions of the Bessel type.

To sum up the situation: prior to 1824 various particular cases of a certain differential equation were investigated by mathematicians, but no attempt was made to deal with these equations in a general way, and the terminology 'Bessel Functions' did not exist.

2. Bessel's coefficients

In 1824 F. W. Bessel investigated a problem associated with elliptic planetary motion. He found that an astronomical quantity, termed the 'eccentric anomaly' θ, could be represented by an infinite series of the form

$$\theta = \chi + A_1 \sin\chi + A_2 \sin 2\chi + \dots \tag{2}$$

$$= \chi + \sum_{n=1}^{\infty} A_n \sin n\chi, \tag{3}$$

where

$$\chi = (\theta - x\sin\theta). \tag{4}$$

The coefficients $A_1,\dots, A_n$ were obtained by a process resembling that used in the Fourier analysis of steady alternating currents. In the latter case, if the cosine terms can be omitted, the wave form is represented by

$$\phi(\theta) = A_0 + A_1 \sin\theta + A_2 \sin 2\theta + \dots. \tag{5}$$

Multiplying both sides by $\sin n\theta$ and integrating between the limits 0 and 2π, we get

$$A_n = \frac{1}{\pi}\int_0^{2\pi} \phi(\theta)\sin n\theta\, d\theta. \tag{6}$$

In the expansion (2) it is found that

$$A_n = \frac{1}{n\pi}\int^{2\pi} \cos n\chi\, d\theta. \tag{7}$$

Since $\chi = \theta - x\sin\theta$, (7) can be written

$$A_n = \frac{1}{n\pi}\int\limits_0^{2\pi} \cos\{n(\theta - x\sin\theta)\}\,d\theta \tag{8}$$

$$= \frac{1}{n\pi}\int\limits_0^{2\pi} \cos(n\theta - z\sin\theta)\,d\theta, \tag{9}$$

where $z = nx$.

If we put $\frac{1}{2}nA_n = J_n(z)$, we obtain Bessel's definition of the functions which bear his name, i.e.

$$J_n(z) = \frac{1}{2\pi}\int\limits_0^{2\pi} \cos(n\theta - z\sin\theta)\,d\theta. \tag{10}$$

By a certain analytical procedure it is possible to evolve from (10) the differential equation

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right)y = 0, \tag{11}$$

where $y = J_n(z)$ and n is integral.† Alternatively it can be shown that $J_n(z)$ is a solution of (11). This generic form is known as Bessel's differential equation. The quantities $\frac{1}{2}nA_n = J_n(z)$ are termed 'Bessel coefficients', but they are more frequently known as Bessel functions of the first kind of order n.

Multiplying both sides of (11) by z^2, we get the equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2)y = 0, \tag{12}$$

whose solution is identical with that of (11). Again, if we write $z = kt$, $\dfrac{dy}{dz} = \dfrac{1}{k}\dfrac{dy}{dt}$, and $\dfrac{d^2y}{dz^2} = \dfrac{1}{k^2}\dfrac{d^2y}{dt^2}$, so (12) becomes

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (k^2t^2 - n^2)y = 0 \tag{13}$$

or

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(k^2 - \frac{n^2}{t^2}\right)y = 0, \tag{14}$$

which is another form of Bessel's equation.

3. Comparison of Bessel and circular functions

Equation (11) can be compared with the familiar equation for the potential difference E in an electrical circuit having inductance L,

† When n is unrestricted the equation takes the same form as (11): see § 4, Chap. IV.

capacity C, and series resistance R (Fig. 1). The p.d. across the condenser is given by

$$\frac{d^2E}{dt^2}+\frac{R}{L}\frac{dE}{dt}+\frac{E}{LC}=0. \tag{15}$$

Since (15) is an equation of the second order in E, it has two independent solutions, that is, they are not proportional. If $R=0$, (15) degenerates to the simple form

$$\frac{d^2E}{dt^2}+\frac{E}{LC}=0, \tag{16}$$

and the two solutions are $A_1\cos\omega t$, $B_1\sin\omega t$, where A_1, B_1 are arbitrary constants, whilst $\omega^2=1/LC$. The complete solution of (16) is, therefore,

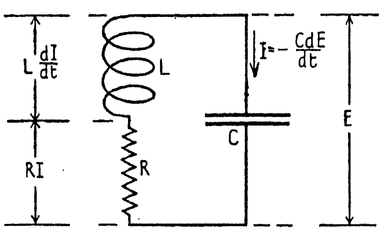


FIG. 1. Diagram of simple electrical oscillatory circuit. The differential equation is obtained from $LDI+RI=E$, by substituting $I=-CDE$ therein ($D=d/dt$).

$$E=A_1\cos\omega t+B_1\sin\omega t. \tag{17}$$

We may, if we please, define the two circular functions $\cos\omega t$, $\sin\omega t$, to be independent solutions of equation (16). $\operatorname{Cos}\omega t$ may be regarded as a circular function of the first kind, whilst $\sin\omega t$ is one of the second kind. This reasoning can be applied to equation (11), so its solution is

$$y=A_1J_n(z)+B_1Y_n(z)=A_1y_1+B_1y_2, \tag{18}$$

where $J_n(z)$, $Y_n(z)$ are Bessel functions of the first and second kinds, respectively, and n is an integer. Consequently we *define* a Bessel function of integral order n to be a solution of equation (11).† As shown above and in § 1, Chap. III, $J_n(z)$ is associated with the coefficients in a certain expansion, so that when n is integral (but not otherwise) $J_n(z)$ can be regarded as either the solution of a differential equation or as a coefficient in an infinite series.

4. First solution of Bessel's equation; functions of zero order

As an illustration of the method of solution, we shall choose the simple case where n in equation (11) is zero. The equation to be solved is then

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+y=0. \tag{19}$$

† If the order is unrestricted the equation takes the same form as (11): see § 4, Chap. IV.

To solve (19) we assume that y can be represented by an infinite power series of the form

$$y = z^m\{a_0+a_1 z+a_2 z^2+a_3 z^3+...\}, \tag{20}$$

where m and the coefficients $a_0, a_1, a_2,...$ are to be determined. This series is to be substituted in (19), and the series obtained by adding the three sets of terms is to be equated to zero. Writing the algebra in suitable form, we have

$$\begin{aligned} y &= z^m\{a_0+a_1 z+a_2 z^2+...\} \\ \frac{1}{z}\frac{dy}{dz} &= z^m\{a_0 mz^{-2}+a_1(m+1)z^{-1}+a_2(m+2)+ \\ &\qquad +a_3(m+3)z+a_4(m+4)z^2+...\} \\ \frac{d^2y}{dz^2} &= z^m\{a_0(m-1)mz^{-2}+a_1 m(m+1)z^{-1}+a_2(m+1)(m+2)+ \\ &\qquad +a_3(m+2)(m+3)z+a_4(m+3)(m+4)z^2+...\}. \end{aligned}$$

If the sum of the three series is to vanish, the sum of the coefficients of like powers of z must also vanish. Equating coefficients to zero, we have $a_0 m^2 = 0$, $a_1(m+1)^2 = 0$, $a_0+a_2(m+2)^2 = 0$, $a_1+a_3(m+3)^2 = 0$, and so on. In the first case $a_0 m^2 = 0$, so either $a_0 = 0$ or $m = 0$. For the present, however, we shall not take either a_0 or m to be zero, as the results obtained in this way will be useful later on. In the second case $a_1(m+1)^2 = 0$, so $a_1 = 0$ provided $m \neq -1$ Similarly $a_3 = a_5 = a_7... = 0$, whilst

$$a_2 = -a_0/(m+2)^2, \qquad a_4 = a_0/(m+2)^2(m+4)^2,$$
$$a_6 = -a_0/(m+2)^2(m+4)^2(m+6)^2,$$

and so on. Inserting the various coefficients in (20) and writing y_1 for y, we get

$$y_1 = a_0 z^m\left\{1-\frac{z^2}{(m+2)^2}+\frac{z^4}{(m+2)^2(m+4)^2}- \right.$$
$$\left. -\frac{z^6}{(m+2)^2(m+4)^2(m+6)^2}+...\right\}. \tag{21}$$

We have seen that if y is to satisfy (19) either a_0 or m must be zero. Hence if we put $m = 0$ and $a_0 = 1$ in (21) we obtain

$$y_1 = J_0(z) = \left\{1-(\tfrac{1}{2}z)^2+\frac{(\frac{1}{2}z)^4}{(2!)^2}-\frac{(\frac{1}{2}z)^6}{(3!)^2}+...\right\} \tag{22}$$

$$= \left\{1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\frac{z^6}{2^2.4^2.6^2}+\ldots\right\} \tag{23}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(\frac{1}{2}z)^{2r}}{(r!)^2}. \tag{24}$$

This series which is absolutely convergent for all values of z, whether real or complex, is the *first* solution of (19), and is defined to be Bessel's function of the *first* kind of order zero. It is denoted by $J_0(z)$ and is identical with (10) when n is zero.

5. Second solution of equation (19)

We have now to find y_2, a second solution of (19), which is linearly independent of y_1. It must be a separate solution and the two solutions have to constitute a fundamental system. If the solutions are linearly dependent, $y_1 = cy_2$ and, therefore, $y_1' = cy_2'$ from which by eliminating c we obtain the Wronskian [93] determinant $W\{y_1, y_2\} = 0$, or $y_1 y_2' - y_2 y_1' = 0$, where $y' = dy/dz$. If the solutions are to be linearly independent, y_1/y_2 cannot be constant, so the requisite condition is that $W\{y_1, y_2\} \neq 0$.

There are various forms of second solution, one of which (due to Neumann) can be found by an extension of the procedure used to determine the first solution. If the series in (21) is substituted in (19) we find that when $a_0 = 1$,

$$\frac{d^2y_1}{dz^2}+\frac{1}{z}\frac{dy_1}{dz}+y_1 = m^2z^{m-2}, \tag{25}$$

which on differentiation with respect to m gives

$$\frac{d^2p}{dz^2}+\frac{1}{z}\frac{dp}{dz}+p = mz^{m-2}(2+m\log z), \tag{26}$$

where $p = \partial y_1/\partial m$.

It follows from (26) that p is a solution of the equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+y = mz^{m-2}(2+m\log z). \tag{26 a}$$

We can obtain $p = \partial y/\partial m$ by differentiating (21) (remembering that y_1 is now y and $a_0 = 1$) since the appropriate conditions are satisfied for $m \geqslant 0$ (see Bromwich, *Infinite Series*). Thus

$$\frac{\partial y}{\partial m} = y_1 \log z + z^m\left\{\frac{z^2}{(m+2)^2}\frac{2}{(m+2)} - \right.$$

$$\left. -\frac{z^4}{(m+2)^2(m+4)^2}\left[\frac{2}{(m+2)}+\frac{2}{(m+4)}\right]+\ldots\right\} \tag{27}$$

Now equation (19), whose second solution we require, is identical with (26 a) when m is zero. Consequently by putting $m = 0$ in (27) we obtain this solution. Thus

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = y_1 \log z + \left\{(\tfrac{1}{2}z)^2 - \frac{(\frac{1}{2}z)^4}{(2!)^2}(1+\tfrac{1}{2}) + \frac{(\frac{1}{2}z)^6}{(3!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \dots\right\}. \tag{28}$$

Since $y_1 = J_0(z)$, it follows that Neumann's Bessel function of the *second* kind of zero order is

$$y_2 = \mathrm{Y}_0(z) = \log z\, J_0(z) + \left\{(\tfrac{1}{2}z)^2 - \frac{(\frac{1}{2}z)^4}{(2!)^2}(1+\tfrac{1}{2}) + \frac{(\frac{1}{2}z)^6}{(3!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \dots\right\} \tag{29}$$

$$= \log z\, J_0(z) - \sum_{r=1}^{\infty} (-1)^r \frac{(\frac{1}{2}z)^{2r}}{(r!)^2}\left\{1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{r}\right\}. \tag{29 a}$$

Another second solution due to Weber is

$$y_2 = Y_0(z) = \frac{2}{\pi}\{\log(\tfrac{1}{2}z)+\gamma\}J_0(z) - \frac{2}{\pi}\sum_{r=1}^{\infty} (-1)^r \frac{(\frac{1}{2}z)^{2r}}{(r!)^2}\left\{1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{r}\right\}, \tag{30}$$

where Euler's constant† $\gamma = 0{\cdot}5772\dots$.

We have now to establish the complete solution of (19) in the form $y = A_1 y_1 + B_1 y_2$, the necessary condition being that $y_1 y_2' - y_2 y_1'$ does not vanish. This is seen to be satisfied by y_1 from (22) taken with y_2 from either (29 a) or (30). Thus the complete solution of (19) is either

$$y = A_1 J_0(z) + B_1 Y_0(z) \tag{31}$$

or

$$y = A_2 J_0(z) + B_2 \mathrm{Y}_0(z). \tag{32}$$

From (29 a) and (30) we find that

$$Y_0(z) = \frac{2}{\pi}\{\mathrm{Y}_0(z) - (\log 2 - \gamma)J_0(z)\}, \tag{33}$$

so that (31) and (32) are identical provided the arbitrary constants are $B_2 = (2/\pi)B_1$ and $A_2 = A_1 - (2/\pi)(\log 2 - \gamma)B_1$, where $(\log 2 - \gamma) = 0{\cdot}1159\dots$. This must be so, for the solution to be unique. The second solution (30) due to Weber will be regarded as standard herein. Consequently the complete solution of equation (19) in standard form is

$$y = A_1 J_0(z) + B_1 Y_0(z). \tag{34}$$

In dealing with numerical values it is immaterial whether $Y_0(z)$ or

† See (8), Chap. IV. $\log_e 2 = 0{\cdot}6931\dots$.

$\mathrm{Y}_0(z)$ is used, but it is preferable to be consistent. A table of values of $Y_0(z)$ is given on page 174; $\mathrm{Y}_0(z)$ is not tabulated.

6. Bessel functions of the third kind of order zero

We have just seen that $J_0(z)$ and $Y_0(z)$ are independent solutions of equation (19). It is useful in certain practical applications to introduce the imaginary i. In certain cases the dependent variable y has two components whose phases differ by $\frac{1}{2}\pi$. Accordingly, Nielsen has defined Bessel functions of the third kind of order zero to be

$$H_0^{(1)}(z) = J_0(z)+iY_0(z) \tag{35}$$

$$H_0^{(2)}(z) = J_0(z)-iY_0(z). \tag{35 a}$$

The letter H is used after Hankel, the German mathematician, and the above are sometimes called Hankelian functions. $H_0^{(1)}(z)$ and $H_0^{(2)}(z)$ are both solutions of (19), and the complete solution with two arbitrary constants is

$$y = A_1 H_0^{(1)}(z)+B_1 H_0^{(2)}(z), \tag{35 b}$$

where A_1 and B_1 are complex.

From (35) and (35 a) by addition and subtraction we obtain

$$J_0(z) = \tfrac{1}{2}\{H_0^{(1)}(z)+H_0^{(2)}(z)\}; \quad Y_0(z) = -\tfrac{1}{2}i\{H_0^{(1)}(z)-H_0^{(2)}(z)\}. \tag{35 c}$$

7. Graphical representation of $J_0(z)$ and $Y_0(z)$

The circular functions of the first and second kinds admit not only of geometrical interpretation but of graphical representation. Cos z is the base of a right-angled triangle in a circular quadrant divided by the radius, whilst it can be exhibited graphically by the well-known curve of period 2π. Although the two Bessel functions $J_0(z)$ and $Y_0(z)$ cannot be interpreted geometrically, they can be represented by curves of slightly variable period. Using Table 1 we can plot the function $J_0(z)$ as shown in curve 1, Fig. 2. It resembles a cosine curve with low damping. The curve commences with $J_0(z) = 1$ when $z = 0$ and crosses the horizontal axis when $z \doteqdot 2{\cdot}405$, $5{\cdot}52$, $8{\cdot}654$, $11{\cdot}792$, $14{\cdot}931$, $18{\cdot}071$, $21{\cdot}212$, $24{\cdot}352$ [77], and so on *ad infinitum*†. The difference between consecutive values of z above $8{\cdot}654$ is approximately π, so the period of $J_0(z)$ is then nearly 2π. These values of z where $J_0(z) = 0$ correspond to the zeros of the series in (22), and are the roots of $J_0(z) = 0$.

† Since $J_0(z)$ is an even function of z, $J_0(z) = J_0(-z)$, so $z \doteqdot -2{\cdot}405$, etc. are also roots, i.e. the curve is symmetrical about the $J_0(z)$ axis.

Using Table 3 we can plot $Y_0(z)$ as shown in curve 2, Fig. 2. At $z = 0$, $Y_0(z)$ is negatively infinite owing to the logarithmic term $J_0(z)\log\frac{1}{2}z$ in (30). Beyond $z = 0\cdot5$ the curve is similar in type to that of $J_0(z)$. The roots of $Y_0(z) = 0$, where the curve crosses the axis of z, are approximately 0·894, 3·958, 7·086, 10·222, 13·361, 16·5, 19·641, and so on *ad infinitum* [93]. Above 10·222, the difference between

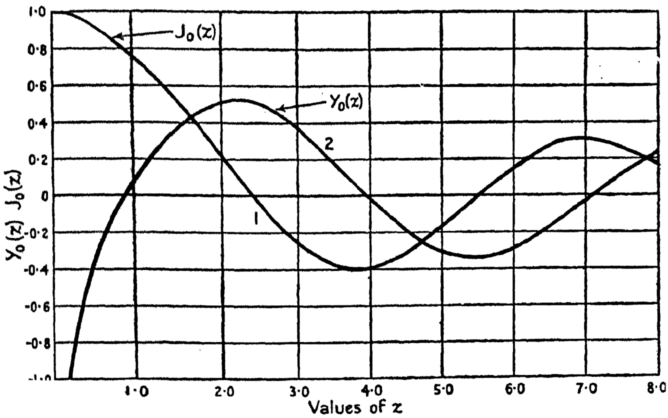


FIG. 2. Graphical representation of Bessel's functions of the first and second kinds of zero order. $Y_0(z)$ is the second solution of (19) as defined by Weber. When $x > 9$ the period of each curve is approximately 2π.

consecutive roots is almost π, so the periods of both $Y_0(z)$ and $J_0(z)$ are approximately 2π.

8. Symmetrical vibrations of a stretched circular membrane

Imagine a very thin piece of drum-skin, aluminium foil, or sheet rubber stretched uniformly over a circular frame. This is known as a membrane and it vibrates when tapped. If the radial tension is large enough, the vibration will be audible. In kettle-drums or tympana,† the point of percussion, i.e. where it is hit, is near the edge and the shape of the membrane during vibration is more complicated than that which we intend to discuss here. Our analytical activities will be confined to vibrations where the shape of the membrane is symmetrical about its polar axis. Thus the dynamic deformation surface is formed by rotating a certain curve about the polar axis. To illus-

† There are various kinds of drum, e.g. kettle-drum of the copper cauldron type, regimental kettle-drum, bass drum, snare drum with wire across a diameter, tom-tom, etc.

trate Bessel's function of order zero, namely (22), we proceed to determine the shape of this curve when the membrane vibrates *in vacuo* at a natural mode without inherent loss. In Fig. 3 A, let $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_v$ be the horizontal and vertical components of the *tangential* radial tension per cm. length, respectively, whilst ξ is the vertical displacement at radius x. Then we have $\dfrac{\boldsymbol{\tau}_v}{\boldsymbol{\tau}} = \dfrac{\partial(\xi_0-\xi)}{\partial x} = -\dfrac{\partial \xi}{\partial x}$, so the total vertical component of the tension on a circle of radius x is $\tau_v = -2\pi x\boldsymbol{\tau}\dfrac{\partial \xi}{\partial x}$

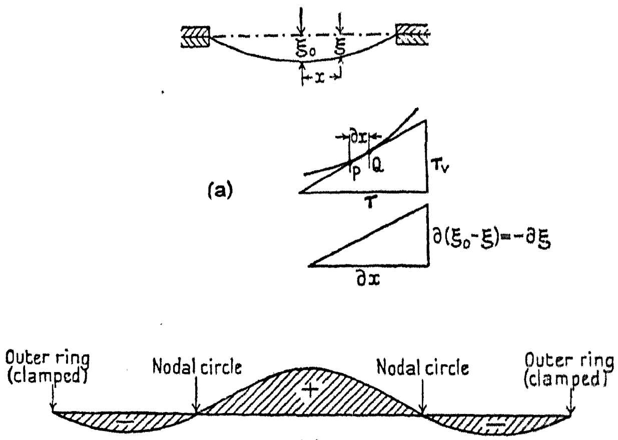


FIG. 3. (*a*) Diagrams relating to vibrating membrane. (*b*) Dynamic deformation curve of stretched circular membrane, devoid of inherent stiffness, vibrating freely *in vacuo* with one nodal circle.

dynes. The net vertical force on the annulus of radial width ∂x is the difference between the vertical tension at P and Q, on the inner and outer edges of the ring. This is found by differentiating τ_v, thus

$$\frac{\partial \tau_v}{\partial x} = -2\pi\boldsymbol{\tau}\left\{x\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial \xi}{\partial x}\right\}, \tag{36}$$

so the net vertical force (elastic) is

$$\partial \tau_v = -2\pi x\boldsymbol{\tau}\,\partial x\left\{\frac{\partial^2 \xi}{\partial x^2}+\frac{1}{x}\frac{\partial \xi}{\partial x}\right\}. \tag{37}$$

Since there is no applied force (the membrane vibrates freely) the sum of the elastic and inertia forces must be zero, so

$$-2\pi x\boldsymbol{\tau}\,\partial x\left(\frac{\partial^2 \xi}{\partial x^2}+\frac{1}{x}\frac{\partial \xi}{\partial x}\right)+2\pi x\,\partial x\,\rho_1\frac{\partial^2 \xi}{\partial t^2} = 0 \tag{38}$$

or
$$\frac{\partial^2\xi}{\partial x^2}+\frac{1}{x}\frac{\partial\xi}{dx}-\frac{\rho_1}{\tau}\frac{\partial^2\xi}{\partial t^2}=0, \tag{39}$$
where ρ_1 is the mass per unit area of the membrane. If the motion is sinusoidal we can write $\xi=\xi_1 e^{i\omega t}$, so $\dfrac{\partial^2\xi}{\partial t^2}=-\omega^2\xi$.

Substituting this value of $\dfrac{\partial^2\xi}{\partial t^2}$ in (39) we obtain
$$\frac{d^2\xi}{dx^2}+\frac{1}{x}\frac{d\xi}{dx}+k_1^2\,\xi=0, \tag{40}$$
where $k_1^2=\rho_1\,\omega^2/\tau$.

Writing $z=k_1 x$, we have
$$\frac{d\xi}{dx}=\frac{d\xi}{dz}\frac{dz}{dx}=k_1\frac{d\xi}{dz}$$
and
$$\frac{d^2\xi}{dx^2}=k_1\frac{d^2\xi}{dz^2}\frac{dz}{dx}=k_1^2\frac{d^2\xi}{dz^2}.$$
When these values of $\dfrac{d\xi}{dx}$ and $\dfrac{d^2\xi}{dx^2}$ are substituted in (40) we obtain
$$\frac{d^2\xi}{dz^2}+\frac{1}{z}\frac{d\xi}{dz}+\xi=0, \tag{41}$$
which is identical in form with (19). Consequently the solution of (41) is
$$\xi=A_1 J_0(z)+B_1 Y_0(z). \tag{42}$$

9. Boundary conditions

The next step in solving the problem of membranal vibrations is to determine the constants A_1 and B_1. To do so we make use of certain conditions associated with the practical side of the problem. At the centre of the membrane $z=k_1 x=0$ and the vibrational amplitude is ξ_0. From a mathematical standpoint this is regarded as a 'boundary condition', although the centre is not a boundary in the ordinary meaning of the term. Inserting the condition $z=0$, $\xi=\xi_0$ in (42) we obtain
$$\xi_0=A_1+B_1(-\infty), \tag{43}$$
since $J_0(0)=1$ and $Y_0(0)=-\infty$ (see Tables or Fig. 2). Now ξ_0 is a small finite amplitude, so we conclude that $B_1=0$ and the second solution $Y_0(z)$ is inadmissible. Thus $A_1=\xi_0$, $B_1=0$ which on insertion in (42) gives
$$\xi=\xi_0 J_0(z)=\xi_0 J_0(k_1 x). \tag{44}$$

There is a second boundary condition to be introduced, since at

the clamping ring where $x = a$, $\xi = 0$. Inserting these values in (44) we get $\xi_0 J_0(k_1 a) = 0$, and since $\xi_0 \neq 0$, it follows that $J_0(k_1 a) = 0$. Accordingly, in the absence of an external or applied driving force, the necessary condition for vibration is that $J_0(k_1 a) = 0$. Moreover, the frequencies of the vibrational modes of the membrane correspond to the roots of $J_0(k_1 a) = 0$. These have been given above as, approximately, $k_1 a = 2{\cdot}405$, $5{\cdot}52$, etc. If $a = 10$ cm. and the radial tension per cm. length at the clamped edge is $\tau = 7\times 10^4$ dynes, and the mass per unit area $\rho_1 = 10^{-2}$ gm., the fundamental frequency of the membrane, corresponding to $k_1 a = 2{\cdot}405$, is $\omega/2\pi = \dfrac{2{\cdot}405\sqrt{\tau}}{2\pi} \doteqdot 100 \sim$.

Its shape during vibration at this frequency is given by $J_0(k_1 x)$, where $k_1 x = z$ varies from 0 to $2{\cdot}405$ as shown in curve 1, Fig. 2. Thus the curve $J_0(z)$ from the origin to its first zero is a semi-cross-section of the membrane by a plane containing the polar axis. The other half-section is identical (since $J_0(z)$ is an even function of z and

$$J_0(z) = J_0(-z)),$$

but it is situated to the left of the vertical axis. The next natural frequency is $100\times 5{\cdot}52/2{\cdot}405 \doteqdot 230 \sim$, and so on. The dynamic deformation curve is now $J_0(z)$ from $z = -5{\cdot}52$ to $+5{\cdot}52$ as shown in Fig. 3 B, there being a nodal circle at $z = k_1 x = 2{\cdot}405$. Its radius is, therefore, $x_1 = 2{\cdot}405/k_1$ and in general $x_n =$ (value of nth root$/k_1$).

10. Membrane driven by constant force f per unit area

When the membrane is driven by a force uniformly distributed over its surface, as in the case of a condenser loud speaker [9, 83], the preceding analysis requires to be modified. Equation (40) gives the natural vibrations of the membrane, whereas we have now to investigate the motion when there is a steady sinusoidal driving force. Considering the forces in operation we have

$$\text{elastic}+\text{inertia} = \text{driving},$$

and (38) now becomes

$$-2\pi x\tau\,\partial x\left(\frac{\partial^2\xi}{\partial x^2}+\frac{1}{x}\frac{\partial\xi}{\partial x}\right)+2\pi x\,\partial x\,\rho_1\frac{\partial^2\xi}{\partial t^2} = \mathbf{f}\,2\pi x\,\partial x,$$

or

$$\frac{d^2\xi}{dx^2}+\frac{1}{x}\frac{d\xi}{dx}+k_1^2\xi = -\frac{\mathbf{f}}{\tau} \tag{45}$$

for harmonic motion. In accordance with the theory of linear differential equations of the second order, the solution of (45) consists of

two parts, (*a*) the complementary function, (*b*) a particular integral. The former is obtained when $\mathbf{f}$ is zero, and as shown above it is $A_1 J_0(k_1 x)+B_1 Y_0(k_1 x)$, the boundary conditions in our case being such that $B_1 = 0$. To find a particular integral we write $\xi = -\mathbf{f}/k_1^2 \boldsymbol{\tau}$, where $\mathbf{f}$ is independent of x. Thus $d\xi/dx$ and $d^2\xi/dx^2$ are both zero so $k_1^2 \xi = -\mathbf{f}/\boldsymbol{\tau}$ and (45) is satisfied. Its complete solution is the sum of the complementary function and the particular integral, so

$$\begin{aligned}\xi &= A_1 J_0(k_1 x)-\mathbf{f}/k_1^2 \boldsymbol{\tau}\\ &= A_1 J_0(k_1 x)-\mathbf{f}/\rho_1 \omega^2. \qquad (46)\end{aligned}$$

To find A_1 we take the boundary condition $\xi = 0$ when $x = a$, this being the radius at the clamped edge of the membrane. Inserting these values in (46) we find that $A_1 J_0(k_1 a) = \mathbf{f}/\rho_1 \omega^2$ or

$$A_1 = \mathbf{f}/\rho_1 \omega^2 J_0(k_1 a). \qquad (47)$$

Substituting the value of A_1 from (47) in (46), the amplitude of vibration of the membrane at a radius x is

$$\xi = \frac{\mathbf{f}}{\rho_1 \omega^2}\left\{\frac{J_0(k_1 x)}{J_0(k_1 a)}-1\right\} = \frac{\mathbf{f}}{\rho_1 \omega^2 J_0(k_1 a)}\{J_0(k_1 x)-J_0(k_1 a)\}. \qquad (48)$$

Also the acceleration is $\ddot{\xi} = -\omega^2 \xi$

$$= -\frac{\mathbf{f}}{\rho_1}\left\{\frac{J_0(k_1 x)}{J_0(k_1 a)}-1\right\}, \qquad (49)$$

whilst the effective mass [83] per unit area, namely,

$$\mathbf{f}/\ddot{\xi} = m_e = -\rho_1 J_0(k_1 a)/[J_0(k_1 x)-J_0(k_1 a)]. \qquad (49\text{a})$$

Since $J_0(k_1 a)$ is constant, for given values of frequency and outer radius, the dynamic deformation curve is determined by

$$\{J_0(k_1 x)-J_0(k_1 a)\}.$$

Now $J_0(k_1 x) = J_0(z)$ is plotted in Fig. 4, and from the ordinates we have to subtract $J_0(k_1 a)$. Knowing the frequency of the driving force, also ρ_1 and $\boldsymbol{\tau}$, we can calculate $k_1 = \omega\sqrt{(\rho_1/\boldsymbol{\tau})}$ and thence $k_1 a$. In Fig. 4 the value of $k_1 a$ is marked off on the horizontal axis, and a vertical line is drawn to meet the curve at P. A horizontal line PQ is drawn from P to meet the vertical axis. Then the shape of the membrane during vibration is given by the ordinates of the shaded portion of the curve, since they represent $\{J_0(k_1 x)-J_0(k_1 a)\}$. The amplitude at R is zero, so there is a nodal circle whose radius is aQR/QP. The membrane is at rest at the outer clamping ring, which corresponds to position P. As shown in § 9, $J_0(k_1 a) = 0$ at a vibrational mode, i.e. a natural frequency. In Fig. 4, $J_0(k_1 a) \neq 0$, so we

see that nodal circles can be obtained on a driven membrane when the frequency does not correspond to that of a free vibration. The amplitude of vibration is, however, relatively small. At a vibrational mode we see from (48) that since $J_0(k_1 a) = 0$, ξ is everywhere infinite (theoretically), except at the clamped edge, where it is zero. This is only to be expected, because the elastic and inertia forces neutralize each other over the surface, and by hypothesis there is neither sound radiation nor inherent mechanical loss.

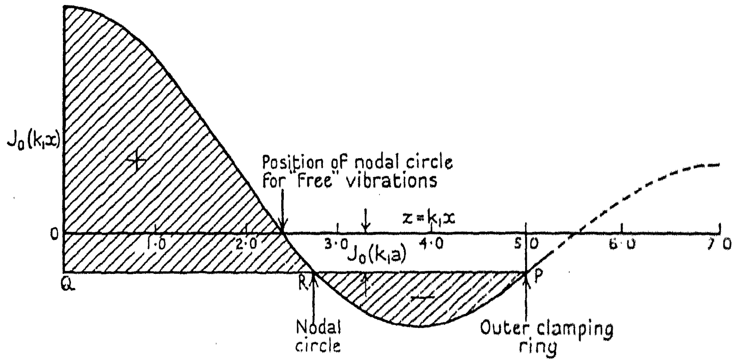


FIG. 4. Dynamic deformation curve of *driven* membrane with one nodal circle, the driving force being uniformly distributed over the surface.

11. Natural symmetrical vibrations of annular membrane

To illustrate the use of the second solution of (19), namely, $Y_0(z)$, the case of an annular membrane clamped between an outer ring of radius a and an inner disk of radius b will be considered. An arrangement resembling this occurs in the hornless moving-coil loud speaker [83]. The membrane will be assumed to vibrate freely without an applied force, i.e. it is set in motion, at one of its modes, by an applied force which is removed after a suitable amplitude is attained.

The dynamical condition at any radius between the inner and outer clamps is similar to that in § 8. Moreover, the analytical work is identical in both cases, apart from choice of the constants A_1 and B_1. Since the membrane is now clamped on *two* circles, the inner boundary condition differs from that of the circular membrane. We have $\xi = 0$ when $x = a$ and also when $x = b$. Inserting these values in (42) we obtain the condition for the natural vibrations to occur, namely:

$$A_1 J_0(k_1 a) + B_1 Y_0(k_1 a) = 0; \quad \text{or} \quad A_1 J_0(k_1 a)/B_1 Y_0(k_1 a) = -1 \qquad (50)$$

$$A_1 J_0(k_1 b) + B_1 Y_0(k_1 b) = 0; \quad \text{or} \quad A_1 J_0(k_1 b)/B_1 Y_0(k_1 b) = -1. \qquad (51)$$

Equating (50) and (51) we find that

$$J_0(k_1 a)Y_0(k_1 b) = J_0(k_1 b)Y_0(k_1 a). \tag{52}$$

The frequencies of the vibrational modes of the annular membrane are found by solving (52) for k_1. This has been treated in detail elsewhere and will not be given here [83], although an approximate method may be of interest. When $k_1 a$ and $k_1 b$ are large enough, we have from formulae (9) and (59) in the final list

$$\begin{aligned} J_0(k_1 b)Y_0(k_1 a)-J_0(k_1 a)Y_0(k_1 b) &\propto \sin(k_1 a-\tfrac{1}{4}\pi)\cos(k_1 b-\tfrac{1}{4}\pi)- \\ &\quad -\sin(k_1 b-\tfrac{1}{4}\pi)\cos(k_1 a-\tfrac{1}{4}\pi) \\ &\propto \sin k_1(a-b). \end{aligned} \tag{53}$$

Now $\sin k_1(a-b) = 0$ when $k_1(a-b) = n\pi$, and since $k_1 = \omega\sqrt{(\rho_1/\tau)}$ we obtain

$$\left(\frac{\tau}{\rho_1}\right)\Big/(a-b), \tag{54}$$

so the frequencies follow the sequence of the natural numbers.

EXAMPLES

1. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+k^2y = 0.$ $[y = A_1 J_0(kz)+B_1 Y_0(kz).]$

2. Solve $z\dfrac{d}{dz}\left(z\dfrac{dy}{dz}\right)+l^2z^2y = 0.$ $[y = A_1 J_0(lz)+B_1 Y_0(lz).]$

3. Solve $\dfrac{d^2v}{dz^2}+\dfrac{1}{z}\dfrac{dv}{dz}+k^2v = a.$ $[v = A_1 J_0(kz)+B_1 Y_0(kz)+a/k^2.]$

4. Solve $\dfrac{d^3y}{dz^3}+\dfrac{1}{z}\dfrac{d^2y}{dz^2}+\left(k^2-\dfrac{1}{z^2}\right)\dfrac{dy}{dz} = 0.$ $\left[\begin{matrix} y = A_1 J_0(kz)+B_1 Y_0(kz)+C_1. \\ D\{D^2y+(1/z)Dy+k^2y\} = 0. \end{matrix}\right]$

5. Show that (a) $zJ_0''(z)+J_0'(z)+zJ_0(z) = 0.$
   (b) $zY_0''(z)+Y_0'(z)+zY_0(z) = 0.$

6. Solve $\dfrac{d^2x}{dz^2}+\dfrac{3}{z}\dfrac{dx}{dz}+\left(k^2+\dfrac{1}{z^2}\right)x = 0.$
   $[x = (1/z)\{A_1 J_0(kz)+B_1 Y_0(kz)\}$. Substitute $x = yz^{-1}$.]

7. (a) Verify that $\dfrac{d}{dx}[xJ_0'(x)] = -xJ_0(x).$
   (b) Solve $d^2y/dt^2-a^2e^{2it}y = 0.$ $[y = A_1 J_0(ae^{it})+B_1 Y_0(ae^{it})$. Substitute $x = ae^{it}$.]

8. Solve $\dfrac{d^2\theta}{dt^2}+\dfrac{1}{t}\dfrac{d\theta}{dt}+4t^2\theta = 0.$ $\Big[\theta = A_1 J_0(t^2)+B_1 Y_0(t^2)$. Substitute $y = t^2$, then $\dfrac{d\theta}{dt} = \dfrac{d\theta}{dy}\dfrac{dy}{dt} = 2t\dfrac{d\theta}{dy}$; $\dfrac{d^2\theta}{dt^2} = 4t^2\dfrac{d^2\theta}{dy^2}+2\dfrac{d\theta}{dy}.\Big]$

9. Solve $z\dfrac{d^2x}{dz^2}+\dfrac{dx}{dz}+\tfrac{1}{4}x = 0.$ $[x = A_1 J_0(z^{\frac{1}{2}})+B_1 Y_0(z^{\frac{1}{2}})$. Substitute $y = z^{\frac{1}{2}}$ and see ex. 8 above.]

10. Solve $\dfrac{d^2v}{dz^2}+\dfrac{1}{z}\dfrac{dv}{dz}+16z^6v = 0$. [$v = A_1 J_0(z^4)+B_1 Y_0(z^4)$. Substitute $x = z^4$.]

11. Solve (a) $y''+\dfrac{1}{z}y'+n^2z^{2n-2}y = 0$. [$y = A_1 J_0(z^n)+B_1 Y_0(z^n)$. Substitute $v = z^n$.]

(b) $z^4u''+e^{2/z}u = 0$. [$u = z\{A_1 J_0(e^{1/z})+B_1 Y_0(e^{1/z})\}$. Substitute $u = yz$, then $v = e^{1/z}$.]

12. Solve $\dfrac{d^2u}{dz^2}+\left(\dfrac{2p+1}{z}\right)\dfrac{du}{dz}+\left(1+\dfrac{p^2}{z^2}\right)u = 0$. [$u = z^{-p}\{A_1 J_0(z)+B_1 Y_0(z)\}$. Substitute $u = vz^{-p}$.]

13. Solve $\dfrac{d^2y}{dz^2}+\left(k^2+\dfrac{1}{4z^2}\right)y = 0$. [$y = z^{\frac{1}{2}}\{A_1 J_0(kz)+B_1 Y_0(kz)\}$. Substitute $y = vz^{\frac{1}{2}}$.]

14. Solve $z^2y''+2zy'+(k^2z^2+\frac{1}{4})y = 0$.
[$y = z^{-\frac{1}{2}}\{A_1 J_0(kz)+B_1 Y_0(kz)\}$. Substitute $y = uz^{-\frac{1}{2}}$.]

15. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+\dfrac{k^2y}{z} = 0$. [$y = A_1 J_0(2kz^{\frac{1}{2}})+B_1 Y_0(2kz^{\frac{1}{2}})$. Substitute $v = z^{\frac{1}{2}}$.]

This is Daniel Bernoulli's equation for the motion of a heavy chain suspended vertically (see § 1).

16. Find a solution of $\dfrac{\partial^2\xi}{\partial x^2}+\dfrac{1}{x}\dfrac{\partial\xi}{\partial x}-\dfrac{\partial^2\xi}{\partial\theta^2} = 0$. [$\xi = e^{in\theta}\{A_1 J_0(nx)+B_1 Y_0(nx)\}$. Substitute $\xi = ye^{in\theta}$, where y is a function of x, but not of θ.]

17. By expanding and integrating term by term, show that

(a) $$J_0(z) = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{iz\cos\theta}\,d\theta = \frac{1}{2\pi}\int\limits_0^{2\pi}\cos(z\cos\theta)\,d\theta = \frac{1}{\pi}\int\limits_0^{\pi}\cos(z\cos\theta)\,d\theta.$$

(b) $$\int\limits_0^{\frac{1}{2}\pi} J_0(z\cos\theta)\cos\theta\,d\theta = \frac{\sin z}{z}.$$

18. By expanding and integrating term by term, show that

$$J_0(z) = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{iz\sin\theta}\,d\theta = \frac{1}{2\pi}\int\limits_0^{2\pi}\cos(z\sin\theta)\,d\theta = \frac{2}{\pi}\int\limits_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\,d\theta.$$

19. By expanding and integrating term by term show that

(a) $$J_0(zi) = \frac{1}{\pi}\int\limits_0^{\pi} e^{z\cos\theta}\,d\theta = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{z\sin\theta}\,d\theta.$$

(b) $$\int\limits_0^{a}\frac{x}{\sqrt{(a^2-x^2)}}J_0(kx\sin\phi)\,dx = a\frac{\sin z}{z},\text{ where } z = ka\sin\phi.$$

[Substitute $x = a\sin\theta$.]

20. Plot the following (a) $y = J_0(z^2)$ from $z = 0$ to 4
(b) $y = J_0(z^{\frac{1}{2}})$ from $z = 0$ to 9
(c) $y = zY_0(z)$ from $z = 0$ to 8
(d) $y = J_0(z)$ from $z = 0$ to -8.

In each case use the tabular values of the functions.

21. Show that $\dfrac{dJ_0(z)}{dz} = J_0'(z) = -\sum_{r=0}^{\infty}(-1)^r\dfrac{(\frac{1}{2}z)^{2r+1}}{r!(r+1)!}$.

22. Solve graphically (*a*) $J_0(z) = z$; (*b*) $\dfrac{4J_0(z)}{z^2} = 1$; (*c*) $Y_0(z)+z = 0$.

[0·83 approx.; 1·46 approx.; 0·48 approx.]

23. Draw the dynamic deformation curve of a circular membrane 10 cm. radius vibrating freely *in vacuo*, when there are two nodal circles and the central amplitude is 0·25 cm. Use suitable vertical and horizontal scales.

24. If, at a vibrational mode, the amplitude of an annular membrane is ξ_0 at a radius x_0, show that

$$\xi = \xi_0\left[\frac{J_0(k_1a)Y_0(k_1x)-Y_0(k_1a)J_0(k_1x)}{J_0(k_1a)Y_0(k_1x_0)-Y_0(k_1a)J_0(k_1x_0)}\right] = \xi_0\left[\frac{J_0(k_1b)Y_0(k_1x)-Y_0(k_1b)J_0(k_1x)}{J_0(k_1b)Y_0(k_1x_0)-Y_0(k_1b)J_0(k_1x_0)}\right].$$

The inner and outer radii of the membrane are b and a.

25. Taking $a = 20$ cm., $b = 15$ cm., $\xi_0 = 0{\cdot}2$ cm. when $x_0 = 18$ cm. and $k_1 = 0{\cdot}2$ in example 24, plot the dynamic deformation curve of the membrane. Make the amplitude scale several times actual size.

26. If (*a*) $Y_0(9{\cdot}5) = 0{\cdot}17121$, find $\mathrm{Y}_0(9{\cdot}5)$. [0·24646.]

(*b*) $\mathrm{Y}_0(5{\cdot}4) = -0{\cdot}53911$, find $Y_0(5{\cdot}4)$. [−0·34017.]

27. What is the least positive value of z for which $Y_0(z) = \mathrm{Y}_0(z)$? Solve graphically. [$z = 0{\cdot}73$.]

28. The solution of a Bessel equation is $y = A_1J_0(z)+B_1Y_0(z)$.

(*a*) If $y = 5$ when $z = 1$, and $y = 1$ when $z = 10$, find A_1 and B_1.

[$A_1 = 2{\cdot}95$, $B_1 = 31$.]

(*b*) Find A_2 and B_2 if $Y_0(z)$ is replaced by $\mathrm{Y}_0(z)$.

[$A_2 = 0{\cdot}666$, $B_2 = 19{\cdot}75$.]

29. The solution of Bessel's equation at (19) on p. 4 is either $y = A_1J_0(z)+B_1Y_0(z)$ or $y = A_2J_0(z)+B_2\mathrm{Y}_0(z)$. Find A_1 and B_1 in terms of A_2 and B_2.

[$A_1 = A_2+B_2(\log 2-\gamma)$; $B_1 = \frac{1}{2}\pi B_2$.]

30. One solution of a Bessel equation is $p = A_1H_0^{(1)}(z)$. If $p = 5(1+i)$ when $z = 1$, find the modulus and the phase of p when $z = 10$. What is the result if $p = A_1H_0^{(2)}(z)$?

[$p = 2{\cdot}31$, $\theta = 565°\,40'$ $(205°\,40')$; $p = 2{\cdot}31$, $\theta = -475°\,40'$ $(-115°\,40')$.]

31. If $P_n(\cos\theta) = 1-\dfrac{n(n+1)}{1^2}\sin^2\tfrac{1}{2}\theta+\dfrac{(n-1)n(n+1)(n+2)}{1^2.2^2}\sin^4\tfrac{1}{2}\theta-\ldots$, show that $\lim_{n\to\infty}P_n\left(\cos\dfrac{z}{n}\right) = J_0(z)$. P_n is Legendre's function of order n (see § 7, Chap. II).

32. A condenser microphone diaphragm is a piece of aluminium alloy foil $2{\cdot}54\times10^{-3}$ cm. thick stretched over a metal framework 4 cm. diameter. What must be the total radial tension if the fundamental frequency *in vacuo* is 5,000 ∼, and the density of the foil is 2·7 gm. cm.$^{-3}$?

[$5{\cdot}9\times10^7$ dynes $= 1{\cdot}33\times10^2$ lb.]

33. Verify that $A_1 J_0(z)+B_1 Y_0(z)+\frac{2}{\pi}\sum_{r=0}^{\infty}(-1)^r z^{2r+1}/[1.3.5...(2r+1)]^2$ is a solution of $y''+\frac{1}{z}y'+y-\frac{2}{\pi z}=0$.

34. Show that

(a) $J_0(z)=\frac{1}{2}[H_0^{(1)}(z)+H_0^{(2)}(z)]$; (b) $Y_0(z)=-\frac{1}{2}i[H_0^{(1)}(z)-H_0^{(2)}(z)]$;

(c) $J_0^2(z)-Y_0^2(z)=\frac{1}{2}[\{H_0^{(1)}(z)\}^2+\{H_0^{(2)}(z)\}^2]$;

(d) $J_0^2(z)+Y_0^2(z)=H_0^{(1)}(z)H_0^{(2)}(z)$.

35. (a) Show that $\int_0^\infty e^{-bx}x^{2n}\,dx=(2n)!/b^{2n+1}$. When a and b are real, use this result to evaluate $\int_0^\infty e^{-bx}J_0(ax)\,dx$.

(b) Show that $\int_0^\infty e^{-az}J_0(bz)\,dz=\left[\frac{1}{a}\int_0^\infty e^{-az}\cos bz\,dz\right]^{\frac{1}{2}}$, and that $\int_0^\infty J_0(z)\,dz=1$.

Interpret the latter result geometrically.

[(a) Expand and integrate term by term. $1/\sqrt{(a^2+b^2)}$.]

36. Show that $\int_0^\infty e^{-bz^2}z^{2n+1}\,dz=n!/2b^{n+1}$. Use this result to verify that $\int_0^\infty ze^{-bz^2}J_0(az)\,dz=e^{-a^2/4b}/2b$.

37. The fundamental frequency of a stretched circular membrane of aluminium foil 20 cm. radius, is 317 $\sim$ *in vacuo*. Plot a curve showing the shape of the membrane when driven by a constant force of 20 dynes per sq. cm. at a frequency of 1,320 $\sim$. Its mass per unit area is 4×10^{-3} gm. Also plot a curve showing the effective mass per unit area at various radii.

38. The spatial distribution of sound due to a narrow annulus of radius a vibrating sinusoidally in an infinite rigid plane, and normally thereto, is given by $p=\frac{\rho_0\ddot{\xi}_0}{r}J_0(ka\sin\phi)$, where ρ_0 is the density of air, $\ddot{\xi}_0$ the axial acceleration of annulus, $r\geqslant 10a$ the distance from the centre to a point in space, ϕ the angular distance of point from the axis, and $k=\omega/c$, c being the velocity of sound $3{\cdot}43\times10^4$ cm. sec.$^{-1}$ Plot polar curves showing the sound distribution on one side of the plane at 100 $\sim$ and 4,000 $\sim$, if $a=10$ cm. [83] This can be done graphically using $J_0(z)$, a sine curve from 0 to $\frac{1}{2}\pi$ and vector radii from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

39. If $J_0'(z)=\frac{d}{dz}J_0(z)=-\sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{2r+1}}{r!(r+1)!}=-J_1(z)$, where $J_1(z)$ is defined to be Bessel's function of unit order, show that $zJ_0(z)=\frac{d}{dz}\{zJ_1(z)\}$, and thence that $zJ_1'(z)=-J_1(z)+zJ_0(z)$.

40. In example 37 the nodal circles on the driven membrane correspond to the roots of $[J_0(k_1 x)-J_0(k_1 a)] = 0$; calculate their radii. Use the curve of $J_0(z)$ if desired. [See Fig. 4.]

41. By differentiating Bessel's equation (19) in § 4 show that

$$(a)\ J_0'''(z)+\frac{1}{z}J_0''+\left(1-\frac{1}{z^2}\right)J_0' = 0;$$

$$(b)\ J_0^{\mathrm{iv}}(z)+\frac{1}{z}J_0'''+\left(1-\frac{2}{z^2}\right)J_0''+\frac{2}{z^3}J_0' = 0.$$ [See example 5 (*a*).]

42. If $J_0(2{\cdot}4) = 0{\cdot}002508$ and $J_0'(2{\cdot}4) = -0{\cdot}520185$, show that a root of $J_0(z)$ occurs at $z = 2{\cdot}40483$.

[Use the first two terms of Taylor's theorem, viz. $f(z+h) \doteqdot f(z)+hf'(z)$ when $h \ll 1$.]

43. If $J_0(z+h) = 0$ and $h \ll 1$, show that $J_0(z)\left\{1-\frac{h^2}{2!}\right\}+J_0'(z)\left\{h-\frac{h^2}{2!z}\right\} = 0.$

[See examples 5 (*a*) and 42.]

II

FUNCTIONS OF HIGHER ORDERS

1. Functions of integral order n

WHEN n is integral,† we have already defined a Bessel function of order n to be a solution of the equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y = 0. \tag{1}$$

The first solution of (1) obtained by an algebraic process similar to that used in Chapter I when n is zero, is found to be

$$a_0\, z^n\left\{1-\frac{(\frac{1}{2}z)^2}{1!(n+1)}+\frac{(\frac{1}{2}z)^4}{2!(n+1)(n+2)}-\frac{(\frac{1}{2}z)^6}{3!(n+1)(n+2)(n+3)}+\ldots\right\}.$$

If we make $a_0 = \dfrac{1}{2^n n!}$ the result is identical with Bessel's function as defined by (10), Chap. I. Thus

$$y_1 = J_n(z)$$

$$= \frac{z^n}{2^n n!}\left\{1-\frac{(\frac{1}{2}z)^2}{1!(n+1)}+\frac{(\frac{1}{2}z)^4}{2!(n+1)(n+2)}-\frac{(\frac{1}{2}z)^6}{3!(n+1)(n+2)(n+3)}+\ldots\right\} \tag{2}$$

$$= (\tfrac{1}{2}z)^n\left\{\frac{1}{n!}-\frac{(\frac{1}{2}z)^2}{1!(n+1)!}+\frac{(\frac{1}{2}z)^4}{2!(n+2)!}-\frac{(\frac{1}{2}z)^6}{3!(n+3)!}+\ldots\right\} \tag{2 a}$$

or

$$J_n(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{n+2r}}{r!(n+r)!}. \tag{3}$$

In Chapter IV, where the gamma or factorial function is introduced, it is shown that‡

$$J_{-n}(z) = (-1)^n J_n(z). \tag{4}$$

We also see from (3) that

$$J_n(-z) = (-1)^n J_n(z), \quad \text{so} \quad J_n(-z) = J_{-n}(z). \tag{4 a}$$

By virtue of (4), $J_{-n}(z)/J_n(z)$ is constant for a given value of n. Moreover, in accordance with § 5, Chap. I, $J_{-n}(z)$ cannot be a second solution of (1), so $J_n(z)$ and $J_{-n}(z)$ do not constitute a fundamental system.

The second solution can be found by a method similar to that used when n is zero. As in that case, various forms have been proposed for this solution, and of those we shall use Weber's, which is denoted by

† In this book n and m will be used to signify integral orders.

‡ Also example 25 at end of Chapter III.

$Y_n(z)$. We have

$$y_2 = Y_n(z) = \frac{2}{\pi}\{\gamma+\log(\tfrac{1}{2}z)\}J_n(z)-\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{2}{z}\right)^{n-2r}-$$

$$-\frac{1}{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{2r+n}}{r!(n+r)!}\left\{1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{r}+1+\frac{1}{2}+\dots+\frac{1}{(n+r)}\right\}. \tag{5}$$

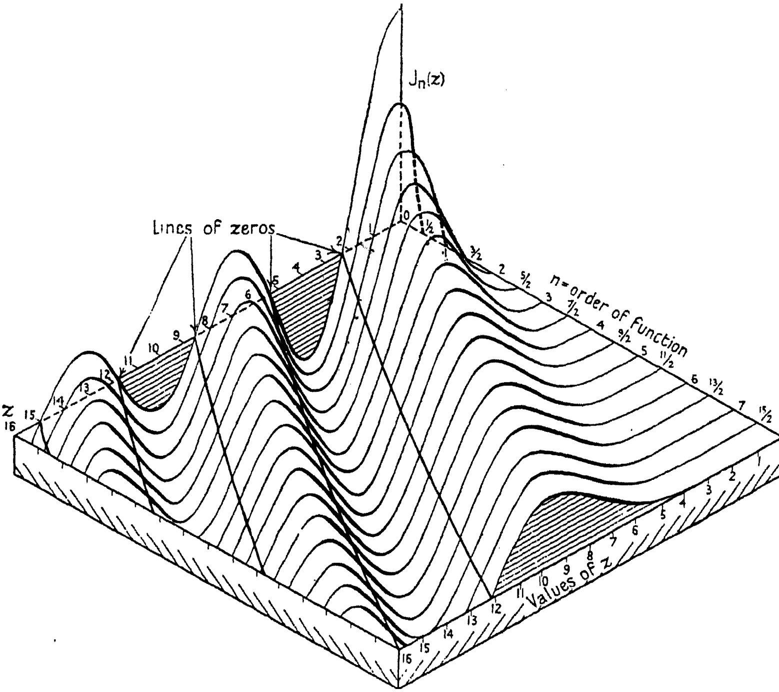


FIG. 5. Isometric plotting of $J_n(z)$.

When r is zero the last bracket is $\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)$.

$Y_n(z)$ and $Y_{-n}(z)$ do not constitute a fundamental system of solutions since

$$Y_{-n}(z) = (-1)^n Y_n(z). \tag{6}$$

Thus the complete solution of equation (1) in standard form is

$$y = A_1 J_n(z)+B_1 Y_n(z). \tag{7}$$

If desired, $Y_n(z)$ can be replaced by $\mathsf{Y}_n(z)$ as in Chapter I when n was zero. The relationship between these two second solutions can be obtained from (33), Chapter I, by changing the order to n.

The function $J_n(z)$ is shown in isometric projection in Fig. 5, the three axes being used for $J_n(z)$, z, and n, respectively. Individual curves represent $J_n(z)$ for certain values of n from 0 to 8 at intervals of $\frac{1}{2}$. This form of projection is best done on isometric graph paper. It is then easy to insert the curves $J_n(z)$ where n is variable and z constant. The addition of the latter curves makes the diagram rather complicated, so they have been omitted from Fig. 5. Since the series for $J_n(z)$ contains the factor $1/(2^n n!)$, the value of the function near the origin diminishes rapidly with increase in n. This is shown clearly on the right-hand side of the diagram.

2. Bessel functions of order ν

When n in equation (1) is *unrestricted*, i.e. it can be any real or complex number, we shall write ν in place of n. Although the values of ν considered throughout the book are real, the majority of the formulae are valid when it is complex. The complete solution of

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{\nu^2}{z^2}\right)y = 0 \tag{8}$$

is found to be

$$y = A_1 J_\nu(z)+B_1 Y_\nu(z), \tag{9}$$

where $J_\nu(z)$ is defined by (16), Chap. IV† and

$$Y_\nu(z) = \frac{\cos\nu\pi\, J_\nu(z)-J_{-\nu}(z)}{\sin\nu\pi}. \tag{10}$$

It can be shown that $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent solutions, thereby constituting a fundamental system. If in (10) ν is integral, $Y_n(z)$ is the limiting value of a fraction. For example, let $\nu \to 1$, then $\cos\nu\pi \to -1$, $\sin\nu\pi \to 0$, $J_\nu(z) \to -J_{-\nu}(z)$ [since $J_{-n}(z) = (-1)^n J_n(z)$], so both the numerator and the denominator of (10) tend to zero. The fraction as a limit represents $Y_1(z)$ when $\nu = 1$. The value of $Y_n(z)$ is found by differentiating the numerator and denominator of (10) and taking the value when $\nu \to n$.

When ν is fractional $\sin\nu\pi$ is not evanescent, so using (10), (9) can be written in the form

$$y = A_2 J_\nu(z)+\frac{B_2}{\sin\nu\pi}\{\cos\nu\pi\, J_\nu(z)-J_{-\nu}(z)\}. \tag{11}$$

Since $\sin\nu\pi$ and $\cos\nu\pi$ are fixed for a definite order ν, we can write‡

$$y = A_1 J_\nu(z)+B_1 J_{-\nu}(z) \tag{11 a}$$

† The definition is not given here because the gamma function has not been treated yet.

‡ $J_{-\nu} \neq (-1)^\nu J_\nu(z)$ unless $\nu = n$ an integer.

as the solution of (8) when ν is non-integral. In (11 a)

$$A_1 = A_2 + B_2 \cot \nu\pi,$$

whilst
$$B_1 = -B_2 \operatorname{cosec} \nu\pi.$$

If (8) is solved by assuming that y can be represented by a power-series of the form given in (20), Chap. I, we obtain for the indicial equation $a_0(m^2-\nu^2) = 0$, and since $a_0 \neq 0$, $m = \pm\nu$. By giving m the values $+\nu$ and $-\nu$ we obtain two linearly independent solutions provided ν is not integral. The solutions in question are $J_\nu(z)$ and $J_{-\nu}(z)$ as in (11 a).

Since $J_\nu(z)$ and $Y_\nu(z)$ are both solutions of (8) for any value of ν, it follows that the functions of the third kind, namely,

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \quad \text{and} \quad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

are also solutions.

The second solution of (8) has been defined in different ways by different mathematicians, although all the solutions are related to each other. To avoid ambiguity Weber's second solution $Y_\nu(z)$ as defined in (10) will be regarded as standard in this book. Its relationship to Neumann's second solution is given by

$$\Upsilon_\nu(z) = \tfrac{1}{2}\pi Y_\nu(z) + (\log 2 - \gamma) J_\nu(z). \tag{12}$$

To recapitulate: Various forms of complete solution of equation (8) are

$$y = A_1 J_\nu(z) + B_1 Y_\nu(z) \qquad \text{always,} \tag{13}$$
$$y = A_1 J_n(z) + B_1 Y_n(z) \qquad n \text{ integral,} \tag{14}$$
$$y = A_1 J_\nu(z) + B_1 J_{-\nu}(z) \qquad \nu \text{ non-integral,}\dagger \tag{15}$$
$$y = A_1 H_\nu^{(1)}(z) + B_1 H_\nu^{(2)}(z) \qquad \text{always,} \tag{16}$$
$$y = A_1 H_n^{(1)}(z) + B_1 H_n^{(2)}(z) \qquad n \text{ integral.} \tag{17}$$

Any of these formulae can be used (except that (14) and (17) are limited to integral orders) in any case, whilst Weber's $Y(z)$ can be replaced by Neumann's $\Upsilon(z)$. The values of the arbitrary constants A_1, B_1 for a given set of boundary conditions will depend upon the formula used. In any particular case the most suitable formula will be that which enables A_1 and B_1 to be determined most easily. The numerical results using tabular values will be the same in all cases.

When the differential equation takes the form

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(k^2 - \frac{\nu^2}{z^2}\right)y = 0,$$

† Under this condition (15) is frequently used in preference to (13).

by writing $x = kz$, we obtain

$$\frac{dy}{dz} = k\frac{dy}{dx} \quad\text{and}\quad \frac{d^2y}{dz^2} = k^2\frac{d^2y}{dx^2}.$$

The equation then becomes

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{\nu^2}{x^2}\right)y = 0,$$

of which the solution is

$$y = A_1 J_\nu(x)+B_1 Y_\nu(x) = A_1 J_\nu(kz)+B_1 Y_\nu(kz). \tag{17 a}$$

The solution may, of course, be written in any of the forms (14) to (17) where the argument is now kz.

Solution (16) is useful when the dependent variable is a physical quantity having two components whose phases differ by $\frac{1}{2}\pi$. For instance, the sound pressure in a loud-speaker horn has two components. The resistive or load component is in phase with the velocity of the air particles, whilst the inertia or wattless component is in quadrature therewith, as illustrated in Fig. 6.

3. Recurrence formulae for $J_n(z)$ and $Y_n(z)$

Using the series in (3), differentiating it and multiplying by z, we obtain

$$z\frac{d}{dz}\{J_n(z)\} = zJ_n'(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(n+2r)(\frac{1}{2}z)^{n+2r}}{r!(n+r)!}$$

$$= nJ_n(z)+z\sum_{r=1}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{n+2r-1}}{(r-1)!(n+r)!}.$$

If in the second term we replace r by $r+1$, we get

$$zJ_n'(z) = nJ_n(z)-z\sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{n+2r+1}}{r!(n+r+1)!}. \tag{18}$$

Thus we have the recurrence formula

$$zJ_n'(z) = nJ_n(z)-zJ_{n+1}(z). \tag{19}$$

By putting $n = 0$, in (19) we find that

$$J_0'(z) = -J_1(z), \tag{20}$$

which is an identity of great utility.

In like manner it can be proved that

$$zJ_n'(z) = -nJ_n(z)+zJ_{n-1}(z). \tag{21}$$

Adding (19) and (21) and dividing throughout by z, we get

$$2J_n'(z) = J_{n-1}(z)-J_{n+1}(z). \tag{22}$$

Subtracting (19) and (21), we have

$$\frac{2n}{z}J_n(z) = J_{n+1}(z)+J_{n-1}(z). \tag{23}$$

Further, $\frac{d}{dz}\{z^{-n}J_n(z)\} = z^{-n}J'_n(z)-nz^{-n-1}J_n(z)$

$= z^{-n-1}\{zJ'_n(z)-nJ_n(z)\} = -z^{-n}J_{n+1}(z)$ from (19).

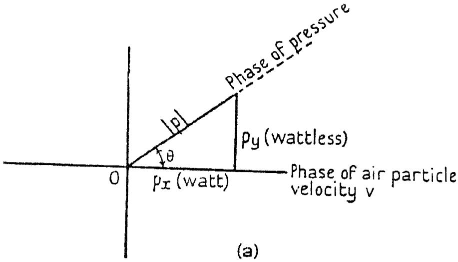
Thus

$$\frac{d}{dz}\{z^{-n}J_n(z)\} = -z^{-n}J_{n+1}(z), \quad \text{or} \quad z^{-n}J_n(z) = -\int^{z}\frac{J_{n+1}(z)\,dz}{z^n}. \tag{24}$$

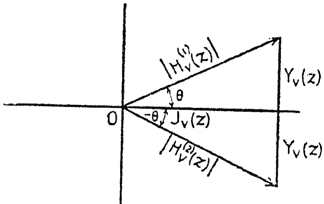
Also $$\frac{d}{dz}\{z^nJ_n(z)\} = z^nJ_{n-1}(z), \quad \text{or} \quad z^nJ_n(z) = \int^{z} z^nJ_{n-1}(z)\,dz. \tag{25}$$

$J'_n(z)$ has been defined as $\frac{d}{dz}J_n(z)$, so if $z = ky$ we must have

$$\frac{d}{d(ky)}J_n(ky) = J'_n(ky), \quad \text{and} \quad \frac{d}{dy}J_n(ky) = kJ'_n(ky).$$



(a)



(b)

FIG. 6. (*a*) Illustrating two components p_x and p_y in quadrature. The sound pressure $p = p_x+ip_y$, the former being in phase with the air particle velocity v, whilst the latter is in quadrature therewith. Since the sound power per unit area is p_xv, p_x is known as the 'watt' or load component. p_y does not contribute anything to the power, so it is the 'wattless' component. (*b*) Vector diagram illustrating Bessel functions of the third kind.

$$H_\nu^{(1)}(z) = J_\nu(z)+iY_\nu(z) = |H_\nu^{(1)}(z)|e^{i\theta},\ H_\nu^{(2)}(z) = J_\nu(z)-iY_\nu(z) = |H_\nu^{(2)}(z)|e^{-i\theta},$$

$$|H_\nu^{(1)}(z)| = |H_\nu^{(2)}(z)| = \sqrt{\{J_\nu^2(z)+Y_\nu^2(z)\}},\ \theta = \tan^{-1}\frac{Y_\nu(z)}{J_\nu(z)}.$$

The recurrence formulae for $Y_n(z)$, $H_n^{(1)}(z)$, $H_n^{(2)}(z)$ are identical in form with those given in (19) to (25). It can be shown that all these relationships are valid when the order is unrestricted, i.e. ν is written for n. Such formulae are of great utility, not only in practical applications of Bessel functions, but in calculating the numerical values of these functions.

4. Example

Solve
$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+4\left(z^2-\frac{n^2}{z^2}\right)y = 0.$$

Substitute $\quad v = z^2 \quad$ and $\quad \dfrac{dy}{dz} = \dfrac{dy}{dv}\dfrac{dv}{dz} = 2z\dfrac{dy}{dv}, \qquad (26)$

$$\frac{d^2y}{dz^2} = 2\frac{d}{dz}\left(z\frac{dy}{dv}\right) = 2\left\{z\frac{d^2y}{dv^2}\frac{dv}{dz}+\frac{dy}{dv}\right\} = 4z^2\frac{d^2y}{dv^2}+2\frac{dy}{dv}. \tag{27}$$

Substituting from (26) and (27) in the original equation, we get

$$4v\frac{d^2y}{dv^2}+4\frac{dy}{dv}+4\left(v-\frac{n^2}{v}\right)y = 0,$$

or
$$\frac{d^2y}{dv^2}+\frac{1}{v}\frac{dy}{dv}+\left(1-\frac{n^2}{v^2}\right)y = 0. \tag{28}$$

The solution of (28) is

$$y = A_1 J_n(v)+B_1 Y_n(v),$$

and, since $v = z^2$, we have

$$y = A_1 J_n(z^2)+B_1 Y_n(z^2). \tag{29}$$

5. Example

The differential equation of a freely vibrating circular membrane for both the symmetrical and radial modes,† is

$$\frac{\partial^2\xi}{\partial x^2}+\frac{1}{x}\frac{\partial\xi}{dx}+\frac{1}{x^2}\frac{\partial^2\xi}{\partial\theta^2}-\frac{1}{b^2}\frac{\partial^2\xi}{\partial t^2} = 0. \tag{30}$$

Substituting $\xi = \chi\sin m\theta\, e^{i\omega t}$, where χ is a function of x alone, we have

$$\frac{\partial\xi}{\partial x} = \frac{\partial\chi}{dx}\sin m\theta\, e^{i\omega t}; \qquad \frac{\partial^2\xi}{\partial x^2} = \frac{d^2\chi}{dx^2}\sin m\theta\, e^{i\omega t};$$

$$\frac{\partial^2\xi}{\partial\theta^2} = -m^2\chi\sin m\theta\, e^{i\omega t}; \qquad \frac{\partial^2\xi}{\partial t^2} = -\omega^2\chi\sin m\theta\, e^{i\omega t}.$$

† It will be seen that when there are no radial modes, i.e. $\dfrac{\partial^2\xi}{\partial\theta^2} = 0$, (30) takes the form of (39), Chap. I, where $b^2 = \tau/\rho_1$.

Inserting these values in (30) we obtain

$$\left\{\frac{d^2\chi}{dx^2}+\frac{1}{x}\frac{d\chi}{dx}+\left(k_m^2-\frac{m^2}{x^2}\right)\chi\right\}\sin m\theta\, e^{i\omega t} = 0, \tag{31}$$

where $k_m^2 = \omega_m^2/b^2$. Accordingly the equation to be solved

$$\frac{d^2\chi}{dx^2}+\frac{1}{x}\frac{d\chi}{dx}+\left(k_m^2-\frac{m^2}{x^2}\right)\chi = 0. \tag{32}$$

As shown below, the only admissible solution of (32), is

$$\chi = A_m J_m(k_m x). \tag{33}$$

Since $\xi = \chi \sin m\theta\, e^{i\omega t}$, we have

$$\xi = A_m e^{i\omega t}\sin m\theta\, J_m(k_m x), \tag{34}$$

where $m = 0, 1, 2,\dots$. It will be seen that a solution can be found if $\sin m\theta$ is replaced by $\cos m\theta$, so $\xi = B_m e^{i\omega t}\cos m\theta\, J_m(k_m x)$ is also a solution of (30). When these solutions are combined we obtain

$$\xi = (A_m \sin m\theta + B_m \cos m\theta)e^{i\omega t}J_m(k_m x) = C_m \cos(m\theta-\epsilon)e^{i\omega t}J_m(k_m x),$$

where $C_m = \sqrt{(A_m^2+B_m^2)}$ and $\epsilon = \tan^{-1}A_m/B_m$.

The solution of (32) involving $Y_m(k_m x)$ is inadmissible since it would make the amplitude ξ infinite at the centre of the membrane where $x = 0$. At the clamped edge of the membrane $x = a$ and $\xi = 0$, so that from (34) we see that $J_m(k_m a) = 0$ is a condition which must always be fulfilled, as in § 9, Chap. I, where only the symmetrical vibrations are in question. Thus the permissible values of $k_m a$ correspond to the roots of $J_m(k_m a) = 0$, and from these the corresponding values of $\omega_m = bk_m$ can be found. The physical interpretation of the solution at (34) is that the membrane vibrates sinusoidally as regards time at a frequency $\omega_m/2\pi$, the amplitude at any radius x being $\xi = A_m \sin m\theta\, J_m(k_m x)$. Starting at the radius $\theta = 0$, the membrane is at rest, a condition which holds when $m\theta = n\pi$, where n is an integer. This means that there are $2m$ radii which are nodal. Choosing a value $\theta = \pi/2m$, the amplitude is $\xi = A_m J_m(k_m x)$ and varies with x according to the value of the Bessel function as in § 9, Chap. I. Consequently, upon a system of nodal circles is superposed a system of nodal radii, but there are frequencies at which the two nodal systems occur independently. The radial system is illustrated in Fig. 7.

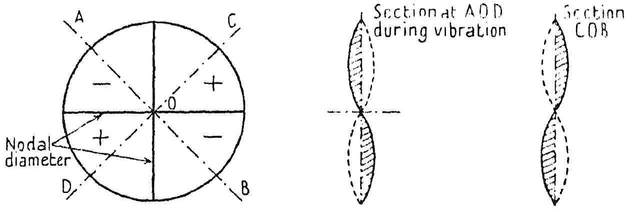


FIG. 7. Illustrating circular membrane vibrating freely *in vacuo* with two nodal diameters. On either side of a nodal line the membrane moves in antiphase as indicated by the signs + and −.

6. Cylinder functions or cylindrical harmonics

If in (30) we write ϕ, r, and z for ξ, x, and t, respectively, and put $b^2 = -1$, the equation becomes

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2} = 0. \tag{35}$$

This is a well-known equation for ϕ, the velocity potential, due to a circular cylinder moving slowly in a fluid. The coordinates r, θ, z are cylindrical polars. Equation (35) was first given by Laplace. Since the Bessel function $J_m(kr)$ is included in the solution of (35), it is known on the Continent as a 'cylindrical harmonic' or as a 'cylinder function'. Following Sonine, the general cylinder function, denoted by the letter $\mathfrak{C}$, is defined to be one which satisfies *both* of the recurrence formulae

$$\mathfrak{C}_{\nu-1}(z)+\mathfrak{C}_{\nu+1}(z) = \frac{2\nu}{z}\mathfrak{C}_\nu(z), \tag{36}$$

$$\mathfrak{C}_{\nu-1}(z)-\mathfrak{C}_{\nu+1}(z) = 2\mathfrak{C}_\nu'(z). \tag{37}$$

By adding and subtracting (36) and (37) two additional recurrence formulae are obtained, these also being satisfied by cylinder functions (see final list of formulae, p. 162). It has already been stated that $J_\nu(z)$ satisfies relationships of the form given by (36) and (37). Since $Y_\nu(z)$, $\mathrm{Y}_\nu(z)$, $H_\nu^{(1)}(z)$, and $H_\nu^{(2)}(z)$ satisfy (36) and (37), they also can be regarded as cylinder functions.

7. Surface zonal (spherical) harmonics

The differential equation of zonal spherical harmonics occurs in the theory of the propagation of sound in a fluid. Since it is of fundamental importance in Acoustical Engineering [83], the solution is

given in full. The equation is

$$r^2\frac{\partial^2\phi}{\partial r^2}+2r\frac{\partial\phi}{\partial r}+k^2r^2\phi+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0. \tag{38}$$

ϕ is the velocity potential (this being a measure of the sound-pressure at a given frequency) at any point in the fluid distant r from the centre of a vibrating spherical surface. $k=\omega/c=2\pi/\lambda$ is the distance phase constant which appears in the phase factor e^{-ikr}, whilst c is the velocity of sound of wave-length λ, and θ is the angular distance of

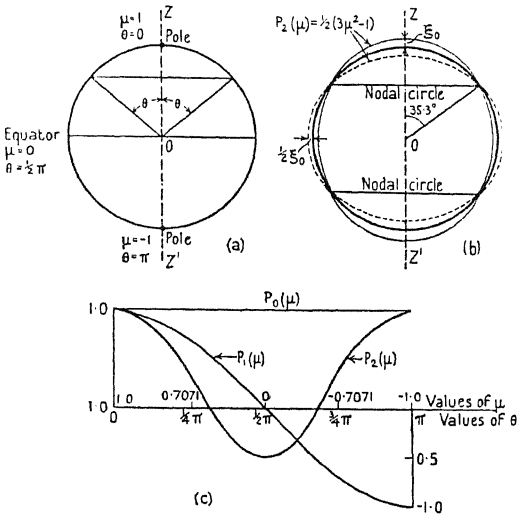


FIG. 8. (a) Illustrating spherical harmonic analysis. (b) Illustrating the surface zonal harmonic $P_2(\mu)$. (c) The surface zonal harmonics $P_0(\mu)$, $P_1(\mu)$, and $P_2(\mu)$ on a linear base. In (b) the phase of the motion is opposite on either side of a nodal circle.

a point on the spherical surface from the pole as shown in Fig. 8. Equation (38) is applicable when the dynamic deformation curve [83] of the sphere is symmetrical about the polar axis ZOZ'.

To solve (38) let $\phi=\xi\chi$, where ξ is a function of θ alone and χ is a function of r alone. Substituting this value of ϕ in (38) and dividing throughout by $\xi\chi$, we obtain

$$\frac{r^2}{\chi}\frac{d^2\chi}{dr^2}+\frac{2r}{\chi}\frac{d\chi}{dr}+k^2r^2=-\frac{1}{\xi\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\xi}{d\theta}\right). \tag{39}$$

Now the three terms on the left in (39) are independent of θ, and since the two sides of the equation are equal it follows that the right-hand

side is also independent of θ, i.e. it is constant. Hence

$$-\frac{1}{\xi\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\xi}{d\theta}\right) = C_1, \tag{40}$$

or
$$\frac{d^2\xi}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{d\xi}{d\theta}+C_1\xi = 0. \tag{41}$$

Substituting $\mu = \cos\theta$, we get

$$\frac{d\xi}{d\theta} = \frac{d\xi}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\frac{d\xi}{d\mu} \tag{42}$$

and
$$\frac{d^2\xi}{d\theta^2} = \sin^2\theta\frac{d^2\xi}{d\mu^2}-\cos\theta\frac{d\xi}{d\mu}. \tag{43}$$

Substituting in (41) from (42), (43), writing $\sin^2\theta = 1-\mu^2$, $2\cos\theta = 2\mu$ and $C_1 = n(n+1)$, we have

$$(1-\mu^2)\frac{d^2\xi}{d\mu^2}-2\mu\frac{d\xi}{d\mu}+n(n+1)\xi = 0. \tag{44}$$

Assume
$$\xi = \mu^m\{a_0+a_1\mu^{-1}+a_2\mu^{-2}+a_3\mu^{-3}...\}. \tag{45}$$

Then

$$\begin{aligned}
n(n+1)\xi &= n(n+1)\mu^m\{a_0 \quad + \quad a_1\mu^{-1} \quad +a_2\mu^{-2} \quad +a_3\mu^{-3}+...\} \\
-2\mu\frac{d\xi}{d\mu} &= \quad -2\mu^m\{ma_0 \quad + \quad (m-1)a_1\mu^{-1} \quad +(m-2)a_2\mu^{-2}+ \\
&\qquad +(m-3)a_3\mu^{-3}+...\} \\
\frac{d^2\xi}{d\mu^2} &= \quad \mu^m\{ \qquad\qquad m(m-1)a_0\mu^{-2}+...\} \\
-\mu^2\frac{d^2\xi}{d\mu^2} &= -\mu^m\{m(m-1)a_0+(m-1)(m-2)a_1\mu^{-1}+ \\
&\qquad +(m-2)(m-3)a_2\mu^{-2}+...\}
\end{aligned}$$

Equating coefficients of μ^m to zero, we get, since $a_0 \neq 0$,

$$m(m-1)+2m-n(n+1) = 0,$$

this being the *indicial* equation from which the value of m is obtained. Thus $(m-n)(m+n+1) = 0$, so $m = n$ or $-(n+1)$. Taking the case where $m = n$, we find on equating coefficients of like powers of μ to zero, that

$$a_1 = a_3... = 0; \quad a_2 = -\frac{n(n-1)}{2(2n-1)}a_0; \quad a_4 = \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}a_0;$$

and so on.

Thus the first solution of (44) is

$$\xi_1 = a_0\left\{\mu^n-\frac{n(n-1)}{2(2n-1)}\mu^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4\,(2n-1)(2n-3)}\mu^{n-4}-...\right\}. \tag{46}$$

If in (46) we put $a_0 = \frac{1.3.5...(2n-1)}{n!}$, we obtain the first Legendre function of order n, namely,

$$\xi_1 = P_n(\mu) = \frac{1.3.5...(2n-1)}{n!}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2}+...\right\}. \quad (47)$$

Thus $P_n(\mu)$ is a solution of (44) and the latter is known as Legendre's equation. If in (47) n is half a positive odd integer, some of the coefficients become infinite, so the first solution no longer exists. In practical work n is usually a positive integer, so (47) is then always a solution of (44), there being a limitation of μ in the present case to the range -1 to $+1$, since $\mu = \cos\theta$ in (41).

TABLE A

Legendre polynomials (functions of the first kind)

Formula	*Order*
$P_0(\mu) = 1$	Zero
$P_1(\mu) = \mu$	Unit
$P_2(\mu) = \frac{1}{2}(3\mu^2-1)$	Second
$P_3(\mu) = \frac{1}{2}(5\mu^3-3\mu)$	Third
$P_4(\mu) = \frac{1}{8}(35\mu^4-30\mu^2+3)$	Fourth
$P_5(\mu) = \frac{1}{8}(63\mu^5-70\mu^3+15\mu)$	Fifth

Polynomials of higher orders can be found either from (47) or from the recurrence formula

$$P_{n+1}(\mu) = \left(\frac{2n+1}{n+1}\right)\mu P_n(\mu) - \left(\frac{n}{n+1}\right)P_{n-1}(\mu)$$
$$(n = 1, 2, 3,...).$$

In finding polynomials of orders higher than those given in the table the relationships $P_n(-1) = (-1)^n$, $P_n(1) = 1$ may be useful.

The series representing the second solution of (44), viz., $Q_n(\mu)$, is obtained by putting $m = -(n+1)$. It is divergent within the range -1 to $+1$ and is therefore inadmissible as a solution of (41). When n is a positive integer the series (47) terminates and we obtain the Legendre polynomials. Putting $n = 0, 1, 2,$ etc. in (47) we get the expressions given in Table A. The polynomials are also the coefficients of corresponding powers of z in the expansion of $\{1-(2\mu z - z^2)\}^{-\frac{1}{2}}$, where $\mu = \cos\theta$ and $(2\mu z - z^2) < 1$. $P_n(\mu)$ is known as a zonal surface harmonic of order n. If we consider the harmonic of unit order, i.e. $P_1(\mu) = \mu$, its value is ± 1 when $\theta = 0$ or π, whilst it is zero when $\theta = \frac{1}{2}\pi$. Thus if we imagine the surface of the sphere to pulsate sinusoidally with radial velocity $u = \mu\cos\omega t = P_1(\mu)\cos\omega t$, there is a nodal circle at the equator. Hence the sphere is divided into two equal zones. Similarly, if the radial velocity is

$$u = \tfrac{1}{2}(3\mu^2-1)\cos\omega t = P_2(\mu)\cos\omega t,$$

there are two nodal circles, symmetrically situated on each side of the equator, at $\mu = \cos\theta = \pm 1/\sqrt{3}$, so the sphere is now divided into three zones. In general, therefore, there are n nodal circles and $(n+1)$ zones corresponding to the Legendre polynomial of order n, and owing to this property of dividing the sphere into zones or belts, these polynomials were called surface zonal harmonics by Thomson and Tait.†

Returning to equation (39), since the right-hand side is equal to C_1, we have

$$r^2\frac{d^2\chi}{dr^2}+2r\frac{d\chi}{dr}+k^2r^2\chi = C_1\chi.$$

From (44), $C_1 = n(n+1)$, so

$$\frac{d^2\chi}{dr^2}+\frac{2}{r}\frac{d\chi}{dr}+\left[k^2-\frac{n(n+1)}{r^2}\right]\chi = 0. \tag{48}$$

On substituting $\chi = r^{-\frac{1}{2}}v$, (48) takes the form

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}+\left[k^2-\frac{(n+\frac{1}{2})^2}{r^2}\right]v = 0,$$

of which the solution is

$$v = A_1 J_{n+\frac{1}{2}}(kr)+B_1 J_{-n-\frac{1}{2}}(kr)$$

or
$$\chi = r^{-\frac{1}{2}}v = \frac{1}{r^{\frac{1}{2}}}\{A_1 J_{n+\frac{1}{2}}(kr)+B_1 J_{-n-\frac{1}{2}}(kr)\}. \tag{49}$$

Thus
$$\phi_n = \xi\chi = \frac{1}{r^{\frac{1}{2}}}\{A_1 J_{n+\frac{1}{2}}(kr)+B_1 J_{-n-\frac{1}{2}}(kr)\}P_n(\mu). \tag{50}$$

This form of the solution is unsuitable for practical requirements so we proceed to find an alternative one. Equation (48) can be written

$$\frac{d^2y}{dr^2}+\left(k^2-\frac{m^2}{r^2}\right)y = 0, \tag{51}$$

where $y = r\chi$ and $m^2 = n(n+1)$. It is desirable to find one solution of (51) in the form We^{-z}, where W is a terminating series or polynomial and $z = ikr$. Putting $y = We^{-z}$, we obtain

$$\frac{dy}{dr} = ike^{-z}\left(\frac{dW}{dz}-W\right); \quad \frac{d^2y}{dr^2} = -k^2e^{-z}\left(\frac{d^2W}{dz^2}-2\frac{dW}{dz}+W\right),$$

so from (51) we have

$$\frac{d^2W}{dz^2}-2\frac{dW}{dz}-\frac{m^2W}{z^2} = 0. \tag{52}$$

† *Natural Philosophy*, 1879.

If we assume that $W = a_0 f_n(z) = a_0+a_1 z^{-1}+a_2 z^{-2}+a_3 z^{-3}+\ldots$ and adopt the same procedure as before, we ultimately find that

$$f_n(z) = \left\{1+\frac{n(n+1)}{2}z^{-1}+\frac{(n-1)n(n+1)(n+2)}{2.4}z^{-2}+\ldots\right\}. \tag{53}$$

Thus
$$r\chi = We^{-z} = a_0\, e^{-ikr} f_n(ikr);$$

so
$$\chi = a_0 \frac{e^{-ikr}}{r} f_n(ikr)$$

and
$$\phi_n = B_n\, \xi\chi = C_n \frac{e^{-ikr}}{r} f_n(ikr) P_n(\mu), \tag{54}$$

a form of solution which is of considerable use in acoustical problems [83], since $f_n(ikr)$ is a terminating series when n is a positive integer, whilst e^{-ikr} is the distance phase factor. (For second solution see [83].)

In (54) if we give n the values 0, 1, 2, 3,..., the corresponding values of $\phi_0, \phi_1, \phi_2,\ldots$ are each separate solutions of (38). Thus we obtain

$$\phi = \sum_{n=0}^{\infty} \phi_n = \frac{e^{-ikr}}{r}\{C_0 P_0(\mu) f_0(ikr)+C_1 P_1(\mu) f_1(ikr)+\ldots\}. \tag{55}$$

During vibration the spherical surface can have any dynamic deformation shape we please, provided it is symmetrical about the polar axis. The radial or normal velocity of the surface (u) will be a function of θ (Fig. 8), and therefore of $\mu = \cos\theta$. The problem is now to determine the coefficients C_n, so that the velocity potential at a point distant r from the centre of the sphere is a function of u.

By definition, the velocity of the air particles vibrating normally to a concentric spherical surface of radius r [83], is $-\frac{\partial \phi}{\partial r}$. At the surface of the vibrator $r = a$, so from (55) its radial velocity is

$$u = \left(-\frac{\partial \phi}{\partial r}\right)_{r=a} = \frac{e^{-ika}}{a^2}\{C_0 P_0(\mu) F_0(ika)+C_1 P_1(\mu) F_1(ika)+\ldots\}, \tag{56}$$

where $F_n(ika) = (1+ika) f_n(ika)-a\left[\frac{d}{dr} f_n(ikr)\right]_{r=a}$.

If we assume that the radial velocity can be written

$$u = \sum_{n=0}^{\infty} u_n = A_0 P_0(\mu)+A_1 P_1(\mu)+\ldots, \tag{57}$$

where $u_n = A_n P_n(\mu)$, it is possible to find A_n. Multiplying both sides of (57) by $P_n(\mu)$ and integrating over the entire sphere (since u applies

to the whole surface), the limits are $\mu = +1$ to -1, i.e. $\theta = 0$ to π (Fig. 8). Thus (see MacRobert, *Spherical Harmonics*)

$$\int_{-1}^{+1} uP_n(\mu)\,d\mu = \int_{-1}^{+1} A_n P_n^2(\mu)\,d\mu = A_n/(n+\tfrac{1}{2}),$$

the other terms vanishing between the limits of integration. Consequently $A_n = (n+\frac{1}{2})\int_{-1}^{+1} uP_n(\mu)\,d\mu$ and this integral can be evaluated when u is known. It will be seen that this procedure of expressing u as a zonal harmonic expansion is akin to resolving an alternating current wave form into a Fourier series. Just as the action of each component of the Fourier series can be considered separately, so also can each spherical harmonic. There is the fundamental difference that in the Fourier case the harmonics relate to different frequencies, whereas in the case of spherical harmonics the question of individual frequencies does not enter. Since the amplitude of the spherical surface is proportional to u, it follows that if the spherical harmonics are added, the resulting curve illustrates the deformation of the surface during vibration. That due to the second harmonic alone is shown in Fig. 8 B.

Since (56) and (57) are equal, so also are the corresponding coefficients of $P_n(\mu)$. Thus $A_n = \dfrac{e^{-ika}}{a^2}C_n F_n(ika)$, so $C_n = \dfrac{a^2e^{ika}}{F_n(ika)}A_n$. Substituting this value of C_n in (55), we obtain

$$\phi = a^2\frac{e^{-ik(r-a)}}{r}\sum_{n=0}^{\infty}\frac{A_n P_n(\mu) f_n(ikr)}{F_n(ika)}$$

$$= a^2\frac{e^{-ik(r-a)}}{r}\sum_{n=0}^{\infty}\frac{u_n f_n(ikr)}{F_n(ika)}, \tag{58}$$

where u_n is the component of the radial velocity of the sphere due to the nth spherical harmonic, and $f_n(ikr)/F_n(ika)$ is a correction factor dependent upon the frequency and the distance from the sphere. At the surface $r = a$, so from (58) the velocity potential there is

$$\phi = a\sum_{n=0}^{\infty}\frac{u_n f_n(ika)}{F_n(ika)}. \tag{59}$$

Owing to $f_n(ika)/F_n(ika)$, the velocity potential has two components, one in phase with the radial velocity and the other in quadrature

therewith. The sound pressure $p = \rho_0 \frac{\partial \phi}{\partial t} = i\rho_0 \omega\phi$ for sinusoidal motion, and it also has two components in quadrature. That in phase with the radial velocity is associated with sound radiation, whilst that in quadrature entails an inertia component due to the reciprocating flow of fluid in the neighbourhood of the sphere. For practical applications in Acoustical Engineering reference can be made to [83].

EXAMPLES

1. Plot $J_1(z)$ from $z = -4$ to $+4$ using Table 2. What is the period approximately?
2. Plot $Y_1(z)$ from $z = 0$ to 8 using Table 4. What is the approximate period of the curve?
3. Plot $J_2(z)$, $J_3(z)$, and $J_4(z)$ from $z = -4$ to $+4$ using Table 5. What are the approximate periods of these curves?
4. Using one of the recurrence formulae and the tables, plot $J_1'(z^2)$.
5. Solve $2zJ_1'(z) = J_1(z)$ graphically, when z lies between 1 and 10.
6. Show that the maximum and minimum values of $J_0(z)$ occur when $J_1(z) = 0$.
7. Plot $J_0(z)$ from $z = 0$ to 1·6, and from the curve find the first four roots of $J_1(z) = 0$ approximately. [3·83; 7·02; 10·2; 13·3.]
8. Show that the maximum and minimum values of $J_n(z)$ occur when

$$(a)\ z = \frac{nJ_n(z)}{J_{n+1}(z)};\quad (b)\ z = \frac{nJ_n(z)}{J_{n-1}(z)};\quad (c)\ J_{n-1}(z) = J_{n+1}(z).$$

9. Establish the following relationships by aid of recurrence formulae:

$$(a)\ J_2(z) = J_0''(z) - \frac{J_0'(z)}{z};\quad (b)\ 2J_0''(z) = J_2(z) - J_0(z);$$

$$(c)\ \frac{J_2(z)}{J_1(z)} = \frac{1}{z} - \frac{J_0''(z)}{J_0'(z)};\quad (d)\ \frac{J_2(z)}{J_1(z)} = \frac{2}{z} - \frac{J_0(z)}{J_1(z)};\quad (e)\ \frac{J_2(z)}{J_1(z)} = \frac{2}{z} + \frac{J_0(z)}{J_0'(z)};$$

$$(f)\ 2J_0'(z)J_2(z) = \left(J_0(z) - J_2(z) - \frac{2J_1(z)}{z}\right)J_1(z).$$

10. Show that

$$(a)\quad z\mathfrak{C}_n'(z) = z\mathfrak{C}_{n-1}(z) - n\mathfrak{C}_n(z);$$

$$(b)\quad z\mathfrak{C}_n'(z) = n\mathfrak{C}_n(z) - z\mathfrak{C}_{n+1}(z);$$

$$(c)\quad \frac{4n\mathfrak{C}_n(z)\mathfrak{C}_n'(z)}{z} = \mathfrak{C}_{n-1}^2(z) - \mathfrak{C}_{n+1}^2(z);$$

(d) $Y_0'(z) = -Y_1(z)$, by differentiating the series for $Y_0(z)$.

11. Show that (a) $zH_n^{(1)\prime}(z) = zH_{n-1}^{(1)}(z) - nH_n^{(1)}(z)$;

(b) $zH_n^{(2)\prime}(z) = nH_n^{(2)}(z) - zH_{n+1}^{(2)}(z)$.

12. If $E_\nu(z) \equiv J_\nu^2(z)$, prove that $E_\nu(z)$ satisfies the recurrence formula

$$E_{\nu-1}(z) - E_{\nu+1}(z) = \frac{2\nu}{z}E_\nu'(z).$$

[Use (22) and (23), also see example 10 (c).]

13. Prove that $\frac{d}{dz}\{\log J_n'(z)\} = [J_{n-2}(z) - 2J_n(z) + J_{n+2}(z)]/2[J_{n-1}(z) - J_{n+1}(z)]$.

14. If θ is any zero of $J_0(z)$ and $h \ll 1$, show that $\frac{J_0(\theta+h)}{h} \doteq -J_1(\theta)$.

[Use Taylor's theorem.]

15. Show that (*a*) $\lim_{z\to 0}\frac{J_n(z)}{z^n} = \frac{1}{2^n n!}$. (*b*) $\lim_{z\to 0}\frac{Y_n(z)}{z^n} = \infty$. (*c*) Prove that the limit ($z \to \infty$) of the bracketed series in (2), § 1 is zero, and thence that when z is finite $\lim_{n\to\infty} J_n(z) = 0$.

[Use the exponential series $e^{-z^2/4(n+1)}$ and treat as an inequality.]

16. If θ is any zero of $J_n(z)$, show that

(*a*) $J_{n-1}(\theta) = -J_{n+1}(\theta)$; (*b*) $2J_{n+1}'(\theta) + J_{n+2}(\theta) = 0$; (*c*) $J_n'(\theta) = J_{n-1}(\theta)$; (*d*) $J_{n-1}'(\theta) + J_n(\theta) = 0$; (*e*) $4J_n''(\theta) = J_{n-2}(\theta) + J_{n+2}(\theta)$.

17. Plot $y = J_0(z)$ from $z = 0$ to 2·405 using Table 1. Plot the parabolic curve $y = 1 - az^2$ which meets the axes at $y = 1$, $z = 2{\cdot}405$. Show that the maximum difference between the ordinates of the curves occurs when $\frac{J_1(z)}{z} \doteq 0{\cdot}346$.

18. The solution of a Bessel equation is $u = A_1 J_n(z) + B_1 Y_n(z)$. When $z = a$, $u = U$, and when $z = b$, $\frac{du}{dz} = 0$. Find B_1.

[$B_1 = U[J_{n+1}(b) - J_{n-1}(b)]/\{J_n(a)[Y_{n-1}(b) - Y_{n+1}(b)] - Y_n(a)[J_{n-1}(b) - J_{n+1}(b)]\}$.]

19. In a problem pertaining to frequency modulation in radio telephony [29], the coefficients a_n are found from the recurrence formula

$$-i(2n\omega\omega_0 + n^2\omega^2)a_n + k\omega_0^2(a_{n-1} - a_{n+1}) = 0.$$

Show that if $\omega \ll \omega_0$, $a_n = i^{-n} J_n\left(\frac{k\omega_0}{\omega}\right)$.

20. In a certain electrical network the currents in the nth, $(n+1)$th and $(n+2)$th sections are connected by the relationship [18]

$$2[\mathbf{Y}_{n+1}\mathbf{Z}_1](n+\nu_0)I_{n+1} - I_n - I_{n+2} = 0,$$

where I is the current and ν_0 an impedance ratio. $\mathbf{Y}_{n+1}$ is the admittance of the $(n+1)$th *shunt* section (see Fig. 17), whilst $\mathbf{Z}_1$ is the *series* impedance of the section corresponding to $n = 1$. Find a formula containing two arbitrary constants, for the current in the $(n+1)$th section.

[$I_{n+1} = A_1 J_{n+\nu_0}(w) + B_1 Y_{n+\nu_0}(w)$ or
$I_{n+1} = A_1 J_{n+\nu_0}(w) + B_1 J_{-n-\nu_0}(w)$, where $w = 1/(\mathbf{Y}_{n+1}\mathbf{Z}_1)$.]

21. Show that $\frac{d}{dx}\{x^{(1+\alpha)/2} J_{(1+\alpha)/(\alpha+\beta+2)}[mx^{(\alpha+\beta+2)/2}]\}$

$$= \frac{m(\alpha+\beta+2)}{2} x^{\alpha}\{x^{(1+\beta)/2} J_{-(1+\beta)/(\alpha+\beta+2)}[mx^{(\alpha+\beta+2)/2}]\}.$$

This is associated with a problem in submarine cable work. See § 5, Chap. VII.

22 (*a*) The values of $J_0(z)$ and $J_1(z)$ for $z = 10$ are, respectively, $-0{\cdot}2459358$ and $0{\cdot}0434727$. Find $J_2(10)$, $J_3(10)$, and $J_4(10)$ without the aid of the tables. [$0{\cdot}254630$; $0{\cdot}058379$; $-0{\cdot}219603$.]

(*b*) The values of $Y_1(z)$ and $Y_0(z)$ for $z = 12$ are $-0{\cdot}0570992$ and $-0{\cdot}2252373$, respectively. Find $Y_2(12)$, $Y_3(12)$, $Y_4(12)$, and $Y_8(12)$ without the aid of tables.

[$0{\cdot}2157208$; $0{\cdot}1290061$; $-0{\cdot}1512177$; $0{\cdot}2614047$.]

23. Solve $\dfrac{d^2u}{dz^2}+\dfrac{1}{z}\dfrac{du}{dz}+4\left(1-\dfrac{1}{z^2}\right)u = 0.$ [$u = A_1 J_2(2z)+B_1 Y_2(2z)$.]

24. Solve $\dfrac{d^2\phi}{dt^2}+\dfrac{1}{t}\dfrac{d\phi}{dt}+5\left(1-\dfrac{\nu^2}{5t^2}\right)\phi = 0.$ [$\phi = A_1 J_n(t\sqrt{5})+B_1 Y_n(t\sqrt{5})$ when $\nu = n$ an integer, or $\phi = A_1 J_\nu(t\sqrt{5})+B_1 J_{-\nu}(t\sqrt{5})$, ν non-integral.]

25. Solve $\dfrac{d^2v}{dt^2}+\dfrac{1}{t}\dfrac{dv}{dt}+3v-\dfrac{v}{4t^2} = 0.$ [$v = A_1 J_{\frac{1}{2}}(t\sqrt{3})+B_1 J_{-\frac{1}{2}}(t\sqrt{3})$.]

26. Solve $\dfrac{d}{dz}\left(z\dfrac{dy}{dz}\right)+k^2zy-\dfrac{\nu^2 y}{z} = 0.$ [$y = A_1 J_n(kz)+B_1 Y_n(kz)$, $\nu = n$ an integer, or $y = A_1 J_\nu(kz)+B_1 J_{-\nu}(kz)$, ν non-integral.]

27. Write down a differential equation one of whose solutions is $y = J_0'(z)$.

$$\left[\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{1}{z^2}\right)y = 0.\right]$$

28. Solve $\dfrac{d^3y}{dz^3}+\dfrac{1}{z}\dfrac{d^2y}{dz^2}+\left(1-\dfrac{\nu^2}{z^2}\right)\dfrac{dy}{dz} = 0$, for y.

$$\left[\frac{dy}{dz} = A_1 J_\nu(z)+B_1 Y_\nu(z);\quad y = \int [A_1 J_\nu(z)+B_1 Y_\nu(z)]\,dz+C_1.\right]$$

29. Solve $\dfrac{d^3y}{dz^3}-\dfrac{1}{z}\dfrac{d^2y}{dz^2}+\left(1+\dfrac{1}{z^2}\right)\dfrac{dy}{dz} = 0.$

$$\left[y = z[A_1 J_1(z)+B_1 Y_1(z)]+C_1.\quad \text{Put } \frac{dy}{dz} = z\phi \text{ and solve for } \phi.\right]$$

30. Show that the equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y = 0,$$

can be written in the form

$$\frac{d^2\varpi}{dz^2}+\left(1-\frac{4n^2-1}{4z^2}\right)\varpi = 0,$$

where $\varpi = yz^{\frac{1}{2}}$.

31. The solution of the equation in example 30 is

$$y = A_1 J_n(z)+B_1 Y_n(z) = A_2 J_n(z)+B_2 \text{Y}_n(z).$$

Show that $A_1 = A_2+B_2(\log 2-\gamma)$ and $B_1 = \frac{1}{2}\pi B_2$.

32. Solve $\dfrac{d^2y}{dz^2}+(2b^2\cosh 2z-\nu^2)y = 0$, when $z > 2$.

[$y = A_1 J_n(2b\cosh z)+B_1 Y_n(2b\cosh z)$ when $\nu = n$ an integer; $y = A_1 J_\nu(2b\cosh z)+B_1 J_{-\nu}(2b\cosh z)$ when ν is non-integral. Take $2\cosh 2z = e^{2z}$ when $z > 2$, and put $x^2 = b^2e^{2z}$.]

33. Solve $\dfrac{d^2y}{dz^2}-(a^2e^{2zi}-n^2)y = 0.$ $[y = A_1 J_n(ae^{iz})+B_1 Y_n(ae^{iz})$. Put $x = ae^{iz}$.]

34. Solve $z^2\dfrac{d^2y}{dz^2}+z\dfrac{dy}{dz}+4(z^4-\nu^2)y = 0.$
[$y = A_1 J_n(z^2)+B_1 Y_n(z^2)$, $\nu = n$ an integer, $y = A_1 J_\nu(z^2)+B_1 J_{-\nu}(z^2)$, ν non-integral.]

35. Solve (a) $z^4u''+(e^{2/z}-\nu^2)u = 0.$
[$u = z\{A_1 J_\nu(e^{1/z})+B_1 J_{-\nu}(e^{1/z})\}$. Substitute $u = yz$, then $v = e^{1/z}$.]

(b) $4\dfrac{d^2\xi}{dt^2}+\dfrac{2}{t}\dfrac{d\xi}{dt}+\dfrac{\xi}{t} = 0.$ [$\xi = t^{\frac{1}{4}}\{A_1 J_{\frac{1}{2}}(t^{\frac{1}{2}})+B_1 J_{-\frac{1}{2}}(t^{\frac{1}{2}})\}$.]

36. Solve $\dfrac{d^2\chi}{d\phi^2}-\dfrac{2\nu}{\phi}\dfrac{d\chi}{d\phi}+k^2\chi = 0.$ [$\chi = \phi^{\nu+\frac{1}{2}}\{A_1 J_{\nu+\frac{1}{2}}(k\phi)+B_1 J_{-\nu-\frac{1}{2}}(k\phi)\}$. Substitute $\chi = y\phi^{-\mu}$ and make $\nu+\mu = -\frac{1}{2}$.]

37. Of what differential equation is $z^{-\frac{1}{2}}J_{\frac{3}{2}}(z)$ a solution?
$$\left[\frac{d^2y}{dz^2}+\frac{2}{z}\frac{dy}{dz}+\left(1-\frac{2}{z^2}\right)y = 0.\right]$$

38. Solve (a) $\dfrac{d^2\phi}{dr^2}+\dfrac{2}{r}\dfrac{d\phi}{dr}+k^2\phi = \dfrac{p^2}{r^2}\phi.$ $\Big[\phi = \dfrac{1}{r^{\frac{1}{2}}}\{A_1 J_\mu(kr)+B_1 J_{-\mu}(kr)\}$, where $\mu^2 = p^2+\frac{1}{4}$. Substitute $\phi = u/r^{\frac{1}{2}}$.$\Big]$

(b) $\dfrac{d^2\phi}{dr^2}-\dfrac{2}{r}\dfrac{d\phi}{dr}+4\left(r^2-\dfrac{1}{r^2}\right)\phi = 0.$ [$\phi = r^{\frac{3}{2}}\{A_1 J_{\frac{5}{4}}(r^2)+B_1 J_{-\frac{5}{4}}(r^2)\}$.]

39. Find the complete solution of $\dfrac{d^2u}{dz^2}\pm\dfrac{u}{z} = 0.$ [$u = z^{\frac{1}{2}}\{A_1 J_1(2z^{\frac{1}{2}})+B_1 Y_1(2z^{\frac{1}{2}})\}$; $u = z^{\frac{1}{2}}\{A_1 J_1(2z^{\frac{1}{2}}i)+B_1 Y_1(2z^{\frac{1}{2}}i)\}$. Substitute $u = xy$, then $x = z^{\frac{1}{2}}$.]

40. Find the complete solution of $\dfrac{d^2v}{dz^2}+z^{-\frac{1}{2}}v = 0.$
[$v = z^{\frac{1}{2}}\{A_1 J_{\frac{2}{3}}(\frac{4}{3}z^{\frac{3}{4}})+B_1 J_{-\frac{2}{3}}(\frac{4}{3}z^{\frac{3}{4}})\}$. Substitute $v = yx^{\frac{2}{3}}$, then $x = z^{\frac{3}{4}}$.]

41. Solve [93] $\dfrac{d^2\varpi}{dz^2}\pm z\varpi = 0.$
[$\varpi = z^{\frac{1}{2}}\{A_1 J_{\frac{1}{3}}(\frac{2}{3}z^{\frac{3}{2}})+B_1 J_{-\frac{1}{3}}(\frac{2}{3}z^{\frac{3}{2}})\}$; $\varpi = z^{\frac{1}{2}}\{A_1 J_{\frac{1}{3}}(\frac{2}{3}z^{\frac{3}{2}}i)+B_1 J_{-\frac{1}{3}}(\frac{2}{3}z^{\frac{3}{2}}i)\}$. Substitute $\varpi = yx^{\frac{1}{3}}$, then $x = z^{\frac{3}{2}}$.]

42. Solve $z\dfrac{d^2\phi}{dz^2}-2\nu\dfrac{d\phi}{dz}-k^2z\phi = 0.$ [$\phi = z^{\nu+\frac{1}{2}}\{J_{\nu+\frac{1}{2}}(kzi)+J_{-\nu-\frac{1}{2}}(kzi)\}$. Substitute $\phi = uz^\mu$ and put $2(\mu-\nu) = 1$.]

43. Solve $y''+\left(\dfrac{2n+1}{z}\right)y'+y = 0.$
[$y = z^{-n}\{A_1 J_n(z)+B_1 Y_n(z)\}$. Substitute $y = vz^{-n}$.]

44. Solve $zy''+(1+n)y'+y = 0.$ [$y = z^{-\frac{1}{2}n}\{A_1 J_n(2z^{\frac{1}{2}})+B_1 Y_n(2z^{\frac{1}{2}})\}$.]

45. Solve [93] $\dfrac{d^2y}{dz^2}+\left(\dfrac{1-\nu}{z}\right)\dfrac{dy}{dz}\pm\dfrac{y}{4z} = 0.$
[$y = z^{\frac{1}{2}\nu}\{A_1 J_\nu(z^{\frac{1}{2}})+B_1 J_{-\nu}(z^{\frac{1}{2}})\}$; $y = z^{\frac{1}{2}\nu}\{A_1 J_\nu(z^{\frac{1}{2}}i)+B_1 J_{-\nu}(z^{\frac{1}{2}}i)\}$.]

46. Solve $\dfrac{d^2y}{dz^2}+\lambda^{2n+2}z^{2n}y = 0.$
[$y = z^{\frac{1}{2}}\{A_1 J_\nu(x)+B_1 J_{-\nu}(x)\}$, where $x = \dfrac{(\lambda z)^{n+1}}{n+1}$ and $\nu = \dfrac{1}{2(n+1)}$. Substitute $y = ux^{1/2m}$, then $x = (\lambda z)^m/m$.]

47. Solve [77] $\dfrac{d^2y}{dz^2}-\left(\dfrac{2\nu\gamma-1}{z}\right)\dfrac{dy}{dz}+\alpha^2\gamma^2z^{2(\gamma-1)}y = 0.$

$[y = z^{\nu\gamma}\{A_1 J_\nu(\alpha z^\gamma)+B_1 Y_\nu(\alpha z^\gamma)\}.]$

48. Transform $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+\left(k^2-\dfrac{n^2}{z^2}\right)y = 0$ to the form $\dfrac{d^2y}{d\theta^2}+(k^2e^{2\theta}-n^2)y = 0.$

49. Find one set of solutions to $\dfrac{\partial^2\phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}+\phi = 0.$

$[\phi = \sum_{n=0}^{\infty}[A_n \sin n\theta\, J_n(r)+B_n\cos n\theta\, J_n(r)]$. Substitute $\phi \doteq y\sin n\theta$, then $y\cos n\theta$, where y is a function of r alone. Then solve for y.]

50. Solve $\dfrac{d^2V}{dx^2}-\dfrac{\alpha}{x}\dfrac{dV}{dx}+k_1^2x^\gamma V = 0.$ $[V = x^p\{A_1 J_\nu(kx^q)+B_1 J_{-\nu}(kx^q)\}$; $q = p/\nu$; $p = \frac{1}{2}(\alpha+1)$; $\nu = \left(\dfrac{\alpha+1}{\gamma+2}\right)$; $k = \dfrac{2k_1}{\gamma+2}$. Substitute $V = yx^p$, then $z = x^q$.]

51. If $Y_{\frac{1}{3}}(10) = 0{\cdot}1702$ and $J_{\frac{1}{3}}(10) = -0{\cdot}1861$, find $\mathrm{Y}_{\frac{1}{3}}(10)$. [0·2459.]

52. If the modulus of $H_{\frac{1}{3}}^{(1)}(14) = 0{\cdot}2132$ and $J_{\frac{1}{3}}(14) = 0{\cdot}2117$, find (a) $Y_{\frac{1}{3}}(14)$; (b) $H_{\frac{1}{3}}^{(2)}(14)$; (c) $\mathrm{Y}_{\frac{1}{3}}(14)$; (d) the phase of $H_{\frac{1}{3}}^{(1)}(14)$.

[(a) 0·025; (b) $0{\cdot}2117-0{\cdot}025i$; (c) 0·064; (d) $-6°\ 46'$. Results by four-figure logarithms.]

53. Show that

$$(a)\ Y_{n+\frac{1}{2}}(z) = (-1)^{n+1}J_{-n-\frac{1}{2}}(z);\quad (b)\ Y_{-n-\frac{1}{2}}(z) = (-1)^nJ_{n+\frac{1}{2}}(z);$$
$$(c)\ H_{n+\frac{1}{2}}^{(1)}(z) = J_{n+\frac{1}{2}}(z)+i(-1)^{n+1}J_{-n-\frac{1}{2}}(z);$$
$$(d)\ H_{n+\frac{1}{2}}^{(2)}(z) = J_{n+\frac{1}{2}}(z)+i(-1)^nJ_{-n-\frac{1}{2}}(z).$$

54. Find an expression for $Y_{-\nu}(z)$ in terms of $J_\nu(z)$ and $J_{-\nu}(z)$. Hence show that

$$(a)\ J_{-\nu}(z) = \frac{\cos\nu\pi\, Y_{-\nu}(z)-Y_\nu(z)}{\sin\nu\pi};\quad (b)\ J_\nu(z) = \frac{Y_{-\nu}(z)-\cos\nu\pi\, Y_\nu(z)}{\sin\nu\pi}.$$

$$\left[Y_{-\nu}(z) = \frac{J_\nu(z)-\cos\nu\pi\, J_{-\nu}(z)}{\sin\nu\pi}.\right]$$

55. Establish the following relationships:

$$(a)\ H_\nu^{(1)}(z) = \frac{J_{-\nu}(z)-e^{-i\nu\pi}J_\nu(z)}{i\sin\nu\pi};\quad (b)\ H_\nu^{(2)}(z) = \frac{e^{i\nu\pi}J_\nu(z)-J_{-\nu}(z)}{i\sin\nu\pi};$$
$$(c)\ H_{-\nu}^{(1)}(z) = e^{i\nu\pi}H_\nu^{(1)}(z);\quad (d)\ H_\nu^{(2)}(z) = e^{i\nu\pi}H_{-\nu}^{(2)}(z);$$
$$(e)\ J_{-\nu}(z) = \tfrac{1}{2}\{e^{i\nu\pi}H_\nu^{(1)}(z)+e^{-i\nu\pi}H_\nu^{(2)}(z)\};$$
$$(f)\ J_{-\nu}(z) = J_\nu(z)\frac{\{e^{i\nu\pi}H_\nu^{(1)}(z)-e^{-i\nu\pi}H_\nu^{(2)}(z)\}}{H_\nu^{(1)}(z)+H_\nu^{(2)}(z)}.$$

56. Show that $(a)\ J_\nu(z) = |H_\nu^{(1)}(z)|\cos\left\{\tan^{-1}\dfrac{Y_\nu(z)}{J_\nu(z)}\right\}$;

$$(b)\ Y_\nu(z) = |H_\nu^{(1)}(z)|\sin\left\{\tan^{-1}\frac{Y_\nu(z)}{J_\nu(z)}\right\}.$$

[See Fig. 6.]

57. Show that $(a)\ J_{-\nu}(z) = \sqrt{\{J_\nu^2(z)+Y_\nu^2(z)\}}\cos(\nu\pi+\theta)$
$= |H_\nu^{(1)}(z)|\cos(\nu\pi+\theta)$;

$(b)\ Y_{-\nu}(z) = |H_\nu^{(1)}(z)|\sin(\nu\pi+\theta).$ $[\theta = \tan^{-1}Y_\nu(z)/J_\nu(z).]$

58. Given $H^{(1)}_{\frac{1}{3}}(6) = 0{\cdot}3254$ with phase $268^\circ\ 7'$ $[\theta = \tan^{-1} Y_{\frac{1}{3}}(6)/J_{\frac{1}{3}}(6)]$, find $J_{\frac{1}{3}}(6)$, $Y_{\frac{1}{3}}(6)$, $J_{-\frac{1}{3}}(6)$, $Y_{-\frac{1}{3}}(6)$. For the two former use the formulae in example (56), and for the two latter the formulae in (57).

[$-0{\cdot}0107$; $-0{\cdot}3253$; $0{\cdot}2763$; $-0{\cdot}1719$. Phase of last two is $268^\circ\ 7' + 60^\circ$.]

59. The radial velocity of a vibrating spherical shell (sinusoidal motion) is $u = U(5\mu^3+9\mu^2+5\mu-3)$. Using the analysis in § 7, show that

$$u = U\{8P_1(\mu)+6P_2(\mu)+2P_3(\mu)\}.$$

Plot u in the form shown in Fig. 8 C.

60. Draw a section of the sphere in (59), by a plane containing the polar axis, showing the dynamic deformation curve when the amplitude of vibration is a maximum. Choose a suitable scale (see Fig. 8 B).

61. What are the values of $f_n(ika)/F_n(ika)$ when (a) $n = 0$; (b) $n = 1$?

[$(1-ika)/(1+k^2a^2)$; $(2+k^2a^2-ik^3a^3)/(4+k^4a^4)$.]

62. Find the expansion of the radial velocity of a sphere, one half of which vibrates axially with root mean square velocity U, the other half being quiescent, in terms of surface zonal harmonics. [$u_0 = \frac{1}{4}U$; $u_1 = \frac{1}{2}\mu U$;

$u_2 = \frac{5}{32}(3\mu^2-1)U$; $u_3 = 0$; $u_4 = -\frac{3}{256}(35\mu^4-30\mu^2+3)U$.

$u = u_0+u_1+\ldots = U\{\frac{1}{4}P_0(\mu)+\frac{1}{2}P_1(\mu)+\frac{5}{16}P_2(\mu)-\frac{3}{32}P_4(\mu)+\ldots\}$.]

63. Find the expansion of the radial velocity of a sphere, one half of which vibrates radially with root mean square velocity U, the other half being quiescent, in terms of surface zonal harmonics.

[$u = u_0+u_1+\ldots = U\{\frac{1}{2}P_0(\mu)+\frac{3}{4}P_1(\mu)-\frac{7}{16}P_3(\mu)+\frac{11}{32}P_5(\mu)-\ldots\}$.]

64. The radial velocity of a sphere of radius a vibrating sinusoidally in air is $u = U$. Show that the surface pressure in phase with the velocity is

$$\rho_0 cU[k^2a^2/(1+k^2a^2)],$$

whilst the quadrature component is

$$\rho_0 cU[ka/(1+k^2a^2)].$$

Hence show that the sound power radiated by a radially pulsating sphere is $4\pi a^2\rho_0 cU^2[k^2a^2/(1+k^2a^2)]$, U being the r.m.s. velocity.

65. Show that the velocity potential at a distance r from a pulsating sphere of radius a is $\phi = \dfrac{Se^{-ik(r-a)}}{4\pi r}$, where S, the strength of the acoustical source, is the velocity-area $UA = 4\pi a^2 U$, U being the root mean square velocity.

III

EXPANSIONS IN TERMS OF BESSEL FUNCTIONS: INTEGRATION

1. (a) The Bessel coefficients

WE have defined $J_n(z)$ to be the first solution of equation (11), Chap. I. Just as the well-known series† for $\cos z$ can be deduced by algebraic processes distinct from the differential equation (16), Chap. I, it is possible to obtain series for $J_n(z)$, apart from equation (11), Chap. I.

Consider the exponential $e^{\frac{1}{2}z(t-1/t)} = e^{\frac{1}{2}zt}\times e^{-\frac{1}{2}z/t}$, when $t \neq 0$. On expansion of the exponentials we obtain

$$\left[1+\tfrac{1}{2}zt+\frac{1}{2!}(\tfrac{1}{2}zt)^2+\frac{1}{3!}(\tfrac{1}{2}zt)^3+...\right]\left[1-\tfrac{1}{2}z/t+\frac{1}{2!}(\tfrac{1}{2}z/t)^2-\frac{1}{3!}(\tfrac{1}{2}z/t)^3+...\right]. \tag{1}$$

When these two absolutely convergent series are multiplied together, there are terms independent of t, i.e. terms involving t^0, terms involving $t, t^2,..., t^n,...$. To select the terms in t^0, multiply corresponding terms in (1), and we get

$$1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\frac{z^6}{2^2.4^2.6^2}+... = J_0(z).$$

Hence $J_0(z)$ is the coefficient of t^0 in the expansion of $e^{\frac{1}{2}z(t-t^{-1})}$. In like manner it can be shown that $J_1(z)$ is the coefficient of t, and in general that $J_n(z)$ is the coefficient‡ of t^n. Inspection of (1) reveals that there are also terms involving $t^{-1}, t^{-2},..., t^{-n}$, and we find $J_{-n}(z)$ to be the coefficient of t^{-n}. Hence the exponential yields an infinite series with Bessel functions of *integral* orders as the *coefficients* of its terms. Thus, when $t \neq 0$,

$$e^{\frac{1}{2}z(t-t^{-1})} = J_0(z)+tJ_1(z)+t^2J_2(z)+... \tag{2}$$
$$+t^{-1}J_{-1}(z)+t^{-2}J_{-2}(z)+...$$
$$= \sum_{n=-\infty}^{+\infty} t^nJ_n(z), \tag{3}$$

n being an integer.

† It is well to recall that the functions sin, cos, etc., are defined by their series and not by the ratios of the sides of a right-angled triangle, which at the best only apply to real quantities.

‡ $e^{\frac{1}{2}z(t-t^{-1})}$ is called the generating function for the Bessel coefficients.

(b) Expansions

If in the exponential index we write $t = e^{i\theta}$, we get

$$e^{\frac{1}{2}z(e^{i\theta}-e^{-i\theta})} = e^{iz\sin\theta}.$$

Substituting this value of t in the right-hand side of (2), we obtain

$$e^{iz\sin\theta} = J_0(z)+\{J_1(z)e^{i\theta}+J_{-1}(z)e^{-i\theta}\}+\{J_2(z)e^{2i\theta}+J_{-2}(z)e^{-2i\theta}\}+\dots. \quad (4)$$

From (4), Chap. II, $J_{-n}(z) = (-1)^n J_n(z)$, so $J_{-1}(z) = -J_1(z)$, $J_{-2}(z) = J_2(z)$, and so on. Substituting these values in (4) we finally derive the expansion

$$e^{iz\sin\theta} = J_0(z)+2\{J_2(z)\cos 2\theta+J_4(z)\cos 4\theta+\dots\}+$$
$$+2i\{J_1(z)\sin\theta+J_3(z)\sin 3\theta+\dots\}. \quad (5)$$

Since $e^{iz\sin\theta} = \cos(z\sin\theta)+i\sin(z\sin\theta)$, we find on equating real and imaginary parts, that

$$\cos(z\sin\theta) = J_0(z)+2\{J_2(z)\cos 2\theta+J_4(z)\cos 4\theta+\dots\} \quad (6)$$

$$= J_0(z)+2\sum_{p=1}^{\infty} J_{2p}(z)\cos 2p\theta. \quad (6\text{ a})$$

$$\sin(z\sin\theta) = 2\{J_1(z)\sin\theta+J_3(z)\sin 3\theta+\dots\} \quad (7)$$

$$= 2\sum_{p=1}^{\infty} J_{2p-1}(z)\sin(2p-1)\theta. \quad (7\text{ a})$$

Multiplying both sides of (6) by $\cos n\theta$ and integrating between the limits 0 and π, all the terms except that involving $\cos^2 n\theta$ vanish, for when m and n are even, $\int_0^{\pi} \cos m\theta\cos n\theta\, d\theta = 0$ unless $m = n$. Since the angles in (6) are even multiples of θ, it follows that

$$\int_0^{\pi} \cos(z\sin\theta)\cos n\theta\, d\theta = \pi J_n(z), \quad (8)$$

when n is an even positive integer. In like manner if both sides of (7) are multiplied by $\sin n\theta$ and the integration performed,

$$\int_0^{\pi} \sin(z\sin\theta)\sin n\theta\, d\theta = \pi J_n(z), \quad (9)$$

provided n is an odd positive integer. Now the integral in (8) vanishes when n is odd, whilst that in (9) vanishes when n is even. It is clear, therefore, that by addition

$$\int_0^{\pi} \{\cos n\theta\cos(z\sin\theta)+\sin n\theta\sin(z\sin\theta)\}\, d\theta = \pi J_n(z), \quad (10)$$

if n is a positive integer.

The integrand in (10) can be abbreviated, so we obtain

$$\int_0^\pi \cos(n\theta - z\sin\theta)\,d\theta = \pi J_n(z), \tag{11}$$

which is identical with Bessel's integral (10), Chap. I, when the interval is 0–π. It will be remembered that (11) is the nucleus from which the general Bessel equation is built up. If we write $\frac{1}{2}\pi - \theta$ for θ (in 5), it is easy to show that

$$e^{iz\cos\theta} = J_0(z) - 2\{J_2(z)\cos 2\theta - J_4(z)\cos 4\theta + \ldots\} + \\ + 2i\{J_1(z)\cos\theta - J_3(z)\cos 3\theta + J_5(z)\cos 5\theta - \ldots\} \tag{12}$$

$$= J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos n\theta. \tag{13}$$

Equating real and imaginary parts

$$\cos(z\cos\theta) = J_0(z) - 2\{J_2(z)\cos 2\theta - J_4(z)\cos 4\theta + J_6(z)\cos 6\theta \ldots\} \tag{14}$$

$$\sin(z\cos\theta) = 2\{J_1(z)\cos\theta - J_3(z)\cos 3\theta + J_5(z)\cos 5\theta \ldots\}. \tag{15}$$

In (14) and (15) put $\theta = 0$, and we get

$$\cos z = J_0(z) - 2\{J_2(z) - J_4(z) + J_6(z) - \ldots\} \tag{16}$$

$$\sin z = 2\{J_1(z) - J_3(z) + J_5(z) - \ldots\}. \tag{17}$$

These formulae show a definite relationship between circular functions and Bessel functions of the first kind. By choosing some suitable value of z and using the tables, it is easy to check (16), (17) numerically.

2. Integration

In physical applications it is frequently necessary to evaluate integrals associated with Bessel functions, so we shall now give some examples.

Evaluate $\int_0^{2\pi} e^{iz\sin\theta}\,d\theta$. The integrand is represented by the series in (5). On integration between the limits 0 and 2π, all the circular functions vanish, since the positive and negative areas are numerically equal. Hence

$$\int_0^{2\pi} e^{iz\sin\theta}\,d\theta = \int_0^{2\pi} J_0(z)\,d\theta = 2\pi J_0(z). \tag{18}$$

Using (12) we get $$\int^{2\pi} e^{iz\cos\theta}\,d\theta = 2\pi J_0(z). \tag{19}$$

Again by (12), since $\cos n\theta$ has the same number of equal positive and negative areas in the interval 0 to π, it follows that

$$\int_0^\pi e^{iz\cos\theta}\,d\theta = \pi J_0(z). \tag{20}$$

In like manner by aid of (13) it is easy to show that

$$J_n(z) = \frac{i^{-n}}{\pi}\int_0^\pi e^{iz\cos\theta}\cos n\theta\,d\theta = \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{iz\cos\theta}\cos n\theta\,d\theta. \tag{21}$$

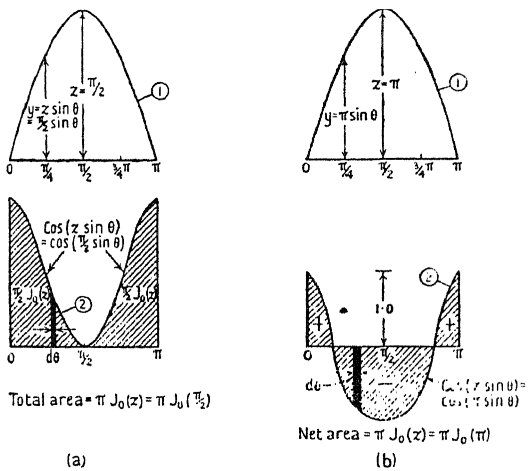


FIG. 9. Graphical representation of $\int_0^\pi \cos(z\sin\theta)\,d\theta$. The elemental areas are each $\cos(z\sin\theta)\,d\theta$.

Also
$$\int_0^\pi \cos(z\sin\theta)\,d\theta = \pi J_0(z), \tag{22}$$

which is seen to be a particular case of (11). This integral can be interpreted geometrically by aid of Fig. 9. Choosing any convenient value for z, $y = z\sin\theta$ is found for θ from 0 to π. These values are plotted in curve 1. $\cos y$ is now obtained and plotted in curve 2. The shaded area is the above integral, i.e. $\pi J_0(z)$, or alternatively $J_0(z)$ is the mean value of $\cos(z\sin\theta)$ over the interval 0 to π. From $\theta = \pi$ to 2π, $\sin\theta$ is negative, but since $\cos y = \cos(-y)$, curve 2 is the same from π to 2π as it is from 0 to π. Thus

$$\int_0^{2\pi} \cos(z\sin\theta)\,d\theta = 2\pi J_0(z).$$

It is to be remarked that in Fig. 9 A, z was chosen to be a fraction of the semi-period of $\sin\theta$. If z were 3·9 say, the diagram would still be symmetrical about $\theta = \frac{1}{2}\pi$.

A similar procedure can be adopted in respect of $\int \sin(z\sin\theta)\,d\theta$ over any interval. From (7) or from graphical processes, we see that over the interval 0 to 2π the integral vanishes.

Now $$\int\limits_0^{2\pi} e^{iz\sin\theta}\,d\theta = \int\limits_0^{2\pi} \cos(z\sin\theta)\,d\theta + i\int\limits_0^{2\pi} \sin(z\sin\theta)\,d\theta$$
$$= 2\pi J_0(z)+0, \tag{23}$$

which means that $\int\limits_0^{2\pi} e^{iz\sin\theta}\,d\theta$ is represented by twice the area of either curve 2 (a) or curve 2 (b) in Fig. 9. By means of rough graphs drawn to show the positive and negative areas, it is frequently possible to deduce the value of this type of integral in an interval which is a fraction of 0 to 2π, provided the result for the whole interval is known.

The original integral can also be evaluated by using the exponential expansion of the integrand. Thus,

$$\int\limits_0^{2\pi} e^{iz\sin\theta}\,d\theta = \int\limits_0^{2\pi} \left[\left(1-\frac{z^2}{2!}\sin^2\theta+\frac{z^4}{4!}\sin^4\theta-\ldots\right)+\right.$$
$$\left.+i\left(z\sin\theta-\frac{z^3}{3!}\sin^3\theta+\ldots\right)\right]d\theta. \tag{24}$$

By aid of rough graphs we see that the integral of the imaginary part vanishes at both the limits. Using the well-known sine reduction formula, viz. $\int\limits_0^{\frac{1}{2}\pi} \sin^{2n}\theta\,d\theta = \dfrac{(2n-1)(2n-3)\ldots1}{2n(2n-2)\ldots2}\,\frac{1}{2}\pi$, we obtain

$$\int\limits_0^{2\pi} e^{iz\sin\theta}\,d\theta = \frac{4\pi}{2}\left\{1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\ldots\right\}, \tag{25}$$

or $$\int\limits_0^{2\pi} \cos(z\sin\theta) = 2\pi J_0(z). \tag{26}$$

In like manner it can be shown that

$$\int\limits_0^{2\pi} e^{-iz\sin\theta}\,d\theta = 2\pi J_0(z). \tag{27}$$

3. Example [5]

Evaluate $\int_0^{2\pi} \sin n\theta\, e^{iz\cos(\theta-\alpha)}\, d\theta$. Using the expansion in (13) the integral becomes

$$\int_0^{2\pi} \sin n\theta \Big\{ J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z) \cos n(\theta-\alpha) \Big\}\, d\theta.$$

The only term which is not evanescent at both the limits of integration is

$$2i^n J_n(z) \int_0^{2\pi} \sin n\theta \cos n(\theta-\alpha)\, d\theta$$

$$= i^n J_n(z) \int_0^{2\pi} [\sin(n\alpha) + \sin n(2\theta-\alpha)]\, d\theta,$$

so
$$\int_0^{2\pi} \sin n\theta\, e^{iz\cos(\theta-\alpha)}\, d\theta = 2\pi i^n J_n(z) \sin n\alpha \tag{28}$$

since the second integral is zero.

4. Example

Evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin^2(nz\cos\theta)}{\sin^2(z\cos\theta)}\, d\theta$. By means of an identity the integrand can be written

$$\frac{\sin^2 ny}{\sin^2 y} = n + 2\{(n-1)\cos y + (n-2)\cos 2y + \ldots \cos(n-1)y\}$$

$$= n + 2\sum_{r=1}^{n-1} (n-r)\cos ry, \tag{29}$$

where $y = z\cos\theta$. For the general term of the integral we get, on omission of $2(n-r)$,

$$\int_0^{\frac{1}{2}\pi} \cos(rz\cos\theta)\, d\theta = \tfrac{1}{2}\pi J_0(rz).$$

Thus
$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2(nz\cos\theta)}{\sin^2(z\cos\theta)}\, d\theta = \tfrac{1}{2}\pi \Big\{ n + 2\sum_{r=1}^{n-1} (n-r) J_0(rz) \Big\}. \tag{30}$$

5. Integration by parts

In evaluating integrals which involve Bessel functions it is frequently necessary to integrate by parts. This process is often used when it is

required to reduce the index of a variable in the integrand at each step. To establish a formula for integrating Bessel functions by this method, we multiply both sides of (3), Chap. II, by z^{n+1} and integrate. Thus

$$\int^{z} z^{n+1}J_n(z)\,dz = 2^{n+1}\int^{z}\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2}z)^{2r+2n+1}}{r!(r+n)!}\,dz \tag{31}$$

$$= z^{n+1}\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2}z)^{2r+n+1}}{r!(r+n+1)!},$$

so
$$\int^{z} z^{n+1}J_n(z)\,dz = z^{n+1}J_{n+1}(z); \tag{32}$$

also
$$\int^{z} z^{n}J_{n-1}(z)\,dz = z^{n}J_n(z). \tag{33}$$

By differentiating (32) and (33) it follows that

$$z^{n+1}J_n(z) = \frac{d}{dz}\{z^{n+1}J_{n+1}(z)\}; \quad \text{or} \quad z^{n+1}J_n(z)\,dz = d\{z^{n+1}J_{n+1}(z)\} \tag{34}$$

and
$$z^{n}J_{n-1}(z) = \frac{d}{dz}\{z^{n}J_n(z)\}; \quad \text{or} \quad z^{n}J_{n-1}(z)\,dz = d\{z^{n}J_n(z)\}. \tag{35}$$

It can be shown that these formulae are valid when the order is unrestricted. Thus n may be replaced by ν.

6. Example

Evaluate
$$\int_0^{z} z^3J_0(z)\,dz.$$

The integral can be written $\int_0^z z^2\{zJ_0(z)\}\,dz$. Putting $n = 0$ in (34) or $n = 1$ in (35) we get $zJ_0(z)\,dz = d\{zJ_1(z)\}$, so the integral becomes

$$\int_0^{z} z^2\,d\{zJ_1(z)\} = z^3J_1(z)-2\int_0^{z} z^2J_1(z)\,dz$$

$$= z^3J_1(z)-2z^2J_2(z) \tag{36}$$

from (32) by putting $n = 1$. The integral $\int_0^z z^mJ_0(z)\,dz$ can always be reduced and evaluated provided m is a positive odd integer. If m is

even, the reduction is terminated by $\int\limits_0^z J_0(z)\,dz$. For example

$$\int\limits_0^z z^2J_0(z)\,dz = \int\limits_0^z z\,d\{zJ_1(z)\}$$

$$= z^2J_1(z)-\int\limits_0^z zJ_1(z)\,dz = z^2J_1(z)+\int\limits_0^z z\,d\{J_0(z)\}$$

$$= z^2J_1(z)+zJ_0(z)-\int\limits^z J_0(z)\,dz. \tag{37}$$

In general, the integral $\int\limits_0^z z^mJ_n(z)\,dz$ can be evaluated if $(m+n)$ is an odd positive integer and $m > n$.

To evaluate $\int\limits_0^z J_n(z)\,dz$ we proceed as follows. Using (22), Chap. II, and writing $(n+m)$ for n, we get

$$J_{n+m-1}(z)-J_{n+m+1}(z) = 2J'_{n+m}(z). \tag{38}$$

Giving m the values $1, 3, 5,...$ successively and integrating, we obtain

$$\int J_n(z)\,dz-\int J_{n+2}(z)\,dz = 2J_{n+1}(z)$$

$$\int J_{n+2}(z)\,dz-\int J_{n+4}(z)\,dz = 2J_{n+3}(z),$$

and so on. By adding both sides separately, we obtain

$$\int\limits_0^z J_n(z)\,dz = 2\{J_{n+1}(z)+J_{n+3}(z)+...\} \tag{39}$$

$$= 2\sum_{m=0}^{\infty} J_{n+2m+1}(z), \tag{40}$$

provided $n > -1$. This restriction is imposed to secure convergence of the integral in (39) at the lower limit $z = 0$. From (2), Chap. II, when $z \to 0$, $J_n(z) = z^n/2^n n!$. Thus $\int J_n(z)\,dz = \dfrac{z^{n+1}}{2^n(n+1)!}$, and when $z \to 0$ this is only convergent if $n+1 > 0$, i.e. if $n > -1$. When n is replaced by ν, the condition is that $R(\nu)$, the real part of ν, > -1.

The integral in (39) can of course be evaluated by expanding $J_n(z)$ and integrating term by term. The series in (39) is useful, however, since sufficient J's can be obtained from tables, provided n is not too

large. As a particular case we have

$$\int_0^z J_0(z)\,dz = 2\{J_1(z)+J_3(z)+...\} = 2\sum_{m=0}^{\infty} J_{2m+1}(z). \tag{41}$$

It can also be shown that (see p. xi)

$$\int_0^z J_0(z)\,dz = zJ_0(z)+\tfrac{1}{2}\pi z\{J_1(z)\mathbf{H}_0(z)-J_0(z)\mathbf{H}_1(z)\}, \tag{42}$$

where $\mathbf{H}_\nu(z)$ is Struve's function which is treated in § 8, Chap. IV. We also have

$$\int_0^z Y_0(z)\,dz = zY_0(z)+\tfrac{1}{2}\pi z\{Y_1(z)\mathbf{H}_0(z)-Y_0(z)\mathbf{H}_1(z)\}. \tag{43}$$

7. Example

Evaluate
$$p = A_1 \int_0^a \int_0^{2\pi} e^{ikx\sin\phi\cos\theta} x\; dxd\theta.$$

The double integral can be written

$$\int_0^a x\,dx \int_0^{2\pi} e^{ikx\sin\phi\cos\theta}\,d\theta. \tag{44}$$

Put $kx\sin\phi = z$ and the first integral in (44) becomes

$$\int_0^{2\pi} e^{iz\cos\theta}\,d\theta = 2\pi J_0(z).$$

Inserting this in (44), substituting $x = z/(k\sin\phi)$ and altering the limits of integration accordingly, we get

$$\frac{2\pi}{(k\sin\phi)^2}\int_0^{ka\sin\phi} zJ_0(z)\,dz = \frac{2\pi}{(k\sin\phi)^2}[zJ_1(z)]_0^{ka\sin\phi}, \tag{45}$$

as from (32) when $n = 0$. Thus

$$p = 2\pi a^2 A_1 \frac{J_1(ka\sin\phi)}{ka\sin\phi}. \tag{46}$$

This represents the spatial sound pressure distribution from one side of a rigid disk vibrating in a plane of infinite extent [4]. The plane is introduced to screen the two sides of the disk from each other. ϕ is the angular distance of any point from the axis of the disk. The linear distance $r \gg a$ the radius. If $\ddot{\xi} = -\omega^2\xi$ is the axial acceleration of the disk, and ρ_0 the density of air, $A_1 = \dfrac{\rho_0\ddot{\xi}}{2\pi r}$, so the pressure

at any spatial point is given by

$$p = \frac{\rho_0 \ddot{\xi} a^2}{\ } \frac{J_1(ka\sin\phi)}{ka\sin\phi}, \tag{47}$$

the negative sign being omitted, since the pressure alternates. The spatial pressure distribution at two typical frequencies for disks 5 cm. and 10 cm. radius is shown in Fig. 10 [83].

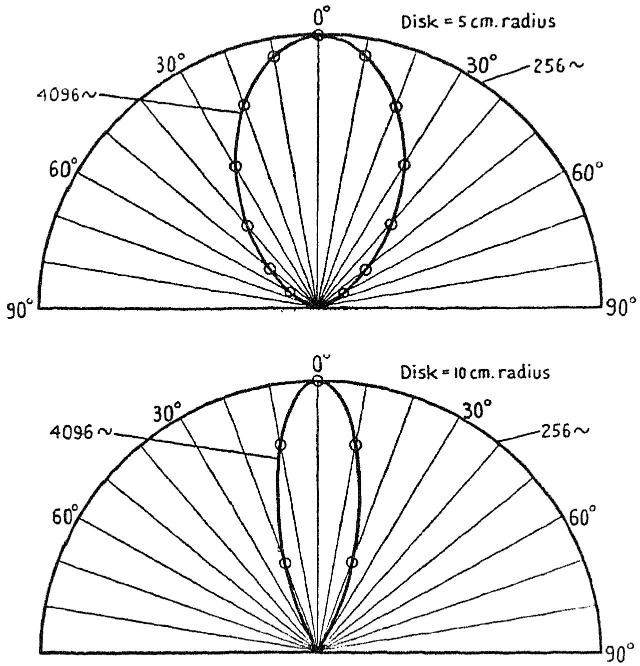


FIG. 10. Polar curves showing the distribution of sound radiation from (one side of) disks 5 cm. and 10 cm. radius vibrating in an infinite rigid plane. The frequencies are 256 ~ (middle of pianoforte) and 4096 ~ (top of pianoforte). $k = \omega$/vel. of sound.

EXAMPLES

1. Prove that

(a) $\cos(z\cosh\theta) = J_0(z) - 2\{J_2(z)\cosh 2\theta - J_4(z)\cosh 4\theta + \ldots\}$.

(b) $\sin(z\cosh\theta) = 2\{J_1(z)\cosh\theta - J_3(z)\cosh 3\theta + \ldots\}$.

[Put $t = ie^{\theta}$ in $e^{\frac{1}{2}z(t-t^{-1})}$.]

2. Prove that

(a) $\sinh(z\cosh\theta) = 2\{I_1(z)\cosh\theta + I_3(z)\cosh 3\theta + \ldots\}$

given† $I_n(z) = i^{-n}J_n(iz)$. [Put $z = zi$ in 1 (b).]

(b) $\cosh(z\cosh\theta) = I_0(z) + 2\{I_2(z)\cosh 2\theta + I_4(z)\cosh 4\theta + \ldots\}$.

[Put $z = zi$ in (1 a).]

† $I_n(z)$ is treated in Chapter VII.

3. Prove that

(a) $\sinh z = 2\{I_1(z)+I_3(z)+...\}$. [Use 2 (a).]

(b) $\cosh z = I_0(z)+2\{I_2(z)+I_4(z)+...\}$. [Use 2 (b).]

(c) $1 = 2\{J_1(\tfrac{1}{2}\pi)-J_3(\tfrac{1}{2}\pi)+J_5(\tfrac{1}{2}\pi)-...\}$.

4. Show that $z = 2\{J_1(z)+3J_3(z)+...\} = 2\sum_{n=0}^{\infty}(2n+1)J_{2n+1}(z)$.

[Differentiate the series for $\sin(z\sin\theta)$ once and put $\theta = 0$.]

5. Show that $z\sin z = 2\{2^2J_2(z)-4^2J_4(z)+...\}$.

[Differentiate the series for $\cos(z\sin\theta)$ twice and put $\theta = \tfrac{1}{2}\pi$.]

6. Show that $z\cos z = 2\{1^2J_1(z)-3^2J_3(z)+...\}$.

[Differentiate the series for $\sin(z\sin\theta)$ twice and put $\theta = \tfrac{1}{2}\pi$.]

7. Prove that $\int_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi}\cos(z\cos\theta)\,d\theta = \tfrac{1}{2}\pi J_0(z)$, and give a geometrical interpretation of the result.

8. Verify that $\int_0^{\frac{1}{2}\pi}\sin(z\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\,d\theta = 2\sum_{n=0}^{\infty}\frac{J_{2n+1}(z)}{2n+1}$.

9. Show that

$$\int_0^{2\pi} e^{\pm z\sin\theta}\,d\theta = 2\pi J_0(zi) = 2\pi I_0(z)\dagger$$
$$= 2\pi\left\{1+\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}+\frac{z^6}{2^2.4^2.6^2}+...\right\}.$$

10. Show that

(a) $$\int_0^{2\pi} e^{iz\cos\theta}\cos n\theta\,d\theta = 2\pi i^n J_n(z).$$

(b) $$\int_0^{2\pi} e^{iz\cos\theta}\sin n\theta\,d\theta = 0.$$

(c) $$\int_0^{2\pi} e^{iz\cos\theta}e^{in\theta}\,d\theta = 2\pi i^n J_n(z).$$

11. Verify that

(a) $$(-1)^{\frac{1}{2}n}\int_0^{\frac{1}{2}\pi}\cos(z\cos\theta)\cos n\theta\,d\theta = \tfrac{1}{2}\pi J_n(z), \qquad n \text{ even}.$$

(b) $$(-1)^{\frac{1}{2}(n-1)}\int_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\cos n\theta\,d\theta = \tfrac{1}{2}\pi J_n(z), \qquad n \text{ odd}.$$

12. Show that $$\int_0^{\frac{1}{2}\pi} J_1(z\cos\theta)\,d\theta = \frac{1-\cos z}{z}.$$

13. Prove that $$\int_0^{\pi}\cos[z\sin(\theta+\alpha)]\,d\theta = \pi J_0(z).$$

† See example 2(a) for definition of $I_n(z)$.

14. Show that [93] $\lim_{n\to\infty} n\log\left\{\cos\left(\frac{z}{n}\right)+i\sin\left(\frac{z}{n}\right)\cos\theta\right\} = iz\cos\theta$, and hence show that when θ lies between 0 and π,

$$J_0(z) = \lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi e^{n\log\{\cos(z/n)+i\sin(z/n)\cos\theta\}}\,d\theta.$$

15. Show that $$\int_0^\pi \sin(z\cos\theta)\sin^{2n}\theta\,d\theta = 0.$$

16. Verify that $\sin(\theta - z\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(z)\sin(\theta+n\phi)$.

17. Evaluate

$$(a)\quad \int_0^\pi e^{r-iaz\cos\theta}\,d\theta. \qquad [\pi e^r J_0(az).]$$

$$(b)\quad \int_0^{\frac{1}{2}\pi} J_0(ka\sin\theta)\sin\theta\cos\theta\,d\theta. \qquad \left[\frac{J_1(ka)}{ka}.\right]$$

18. Prove that $$\int_0^z J_0(z)\,dz = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin(z\sin\theta)\,d\theta}{\sin\theta}.$$

[Substitute $(2/\pi)\int_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\,d\theta$ for $J_0(z)$ and alter the order of integration. The result is $\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}d\theta\int_0^z\frac{\cos(z\sin\theta)}{\sin\theta}d(z\sin\theta)$.]

19. Show that $\int_0^z J_n(z)\,dz = \frac{1}{\pi}\int_0^\pi\frac{\sin(z\sin\theta)}{\sin\theta}\cos n\theta\,d\theta$, when n is even and positive. [See example 18.]

20. Using example 18, and given that $\int_{0}^{\frac{1}{2}\pi}\frac{\sin n\theta\,d\theta}{\sin\theta} = \frac{1}{2}\pi$ when n is odd, prove that $\int_0^z J_0(z)\,dz = 2\sum_{n=0}^{\infty}J_{2n+1}(z)$. Also prove that $\int_0^{\frac{1}{2}\pi}\frac{\sin n\theta\,d\theta}{\sin\theta} = \frac{1}{2}\pi$.

21. Evaluate $\int_0^\pi\cos(z\cos\theta)\sin^2\theta\,d\theta$, and give a graphical representation of the integration when $z = 5$. Check the result of the integration by a graphical method. $\left[\pi\frac{J_1(z)}{z} = -0{\cdot}206 \text{ when } z = 5.\right]$

22. Prove that
$$\frac{z^n}{1.3.5...(2n-1)}\int_0^\pi\cos(z\cos\theta)\sin^{2n}\theta\,d\theta = \pi J_n(z)$$
$$= \pi\sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{2r+n}}{r!(r+n)!}.$$

23. Prove that $$\int^\pi e^{-iz\sin\theta}e^{in\theta}\,d\theta = 2\pi J_n(z).$$

24. From (3) in the text $e^{\frac{1}{2}z(t-t^{-1})} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$. By differentiating both sides [93] and equating coefficients of t^n show that $J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z)$.

25. By writing $-t^{-1}$ for t in (24), show that $\sum_{n=-\infty}^{\infty} (-t)^{-n} J_n(z) = \sum_{n=-\infty}^{\infty} t^n J_n(z)$.

By equating coefficients of t^n, show that

$$(-1)^n J_{-n}(z) = J_n(z) \quad \text{i.e. } J_{-n}(z) = (-1)^n J_n(z).$$

26. Show that $\displaystyle\sum_{r=1}^{n} \int_0^{\frac{1}{2}\pi} \frac{\sin\{(2r-1)z\cos\theta\}\,d\theta}{\sin(z\cos\theta)} = \tfrac{1}{2}\pi\Big\{n + 2\sum_{r=1}^{n-1}(n-r)J_0(z)\Big\}.$

[Use formula for sines in A.P. and example § 4 in the text.]

27. Evaluate (*a*) $\displaystyle\int_0^a zJ_0(kz)\,dz$; (*b*) $\displaystyle\int_0^1 J_0(z)\,dz$; (*c*) $\displaystyle\int_0^1 Y_0(z)\,dz$; (*d*) $\displaystyle\int_0^1 r^{\frac{3}{2}} J_{\frac{1}{2}}(ra)\,dr$.

$$\left[a^2 \frac{J_1(ka)}{ka};\ 0{\cdot}92;\ -0{\cdot}637;\ \frac{1}{a} J_{\frac{3}{2}}(a).\right]$$

28. Show that
$$\int^z zJ_1(kz)\,dz = -\frac{z}{k}J_0(kz) + \frac{1}{k}\int^z J_0(kz)\,dz$$
$$= -\frac{z}{k}J_0(kz) + \frac{2}{k^2}\sum_{n=0}^{\infty} J_{2n+1}(kz).$$

29. Evaluate
$$(a)\ \int_0^a z^5 J_2(z)\,dz; \quad (b)\ \int^x xJ_{\frac{2}{3}}(x^{\frac{3}{2}})\,dx; \quad (c)\ \int^x xJ_{-\frac{2}{3}}(x^{\frac{3}{2}})\,dx.$$

[(*a*) $a^5 J_3(a) - 2a^4 J_4(a)$; (*b*) $-\frac{2}{3}x^{\frac{1}{2}} J_{-\frac{1}{3}}(x^{\frac{3}{2}})$; (*c*) $\frac{2}{3}x^{\frac{1}{2}} J_{\frac{1}{3}}(x^{\frac{3}{2}})$.
Put $x = v^{\frac{2}{3}}$ and use formulae (24) and (25), Chap. II.]

What does the integration in (*a*) signify geometrically?

30. Evaluate [6] the following integral which occurs in acoustical work:

$$\int_0^a \frac{x^3}{a^2}\,dx \int_0^{2\pi} e^{ikx\sin\phi\cos\theta}\,d\theta.$$

$$\left[2\pi a^2\left(\frac{J_1(ka\sin\phi)}{ka\sin\phi} - 2\frac{J_2(ka\sin\phi)}{(ka\sin\phi)^2}\right).\right]$$

31. Evaluate [6]
$$\int_0^a\int_0^{2\pi}\left(1-\varphi_1\frac{r^2}{a^2}\right)\left(1-\varphi_2\frac{r^2}{a^2}\right) re^{ikr\sin\phi\cos\theta}\,dr\,d\theta.$$

This integral occurs in finding the distribution of sound from a disk vibrating with two nodal circles.

$$\left[2\pi a^2\left\{[1-(\varphi_1+\varphi_2)+\varphi_1\varphi_2]\frac{J_1(z)}{z} + 2[(\varphi_1+\varphi_2)-2\varphi_1\varphi_2]\frac{J_2(z)}{z^2} + 8\varphi_1\varphi_2\frac{J_3(z)}{z^3}\right\},\right.$$
$$\left.\text{where } z = ka\sin\phi.\right]$$

32. Evaluate [6] $\int_0^a \int_0^{2\pi} \left(\frac{r}{a}\right)^n \sin n\theta \, e^{ikr\sin\phi\cos(\theta-\alpha)} r \, dr d\theta.$

This integral occurs in finding the sound distribution from a disk vibrating with n nodal diameters.

$$\left[2\pi i^n a^2 \sin n\alpha\left(\frac{J_{n+1}(ka\sin\phi)}{ka\sin\phi}\right).\right]$$

33. Evaluate $\int_0^z z^m J_0(z)\,dz$ when m is an odd positive integer.

$$\Big[J_1\{z^m-(m-1)^2z^{m-2}+(m-1)^2(m-3)^2z^{m-4}-\ldots\}+$$
$$+J_0\{(m-1)z^{m-1}-(m-1)^2(m-3)z^{m-3}+\ldots\}$$
$$\text{or}\quad z^mJ_1-(m-1)z^{m-1}J_2+\ldots\{(m-1)(m-2)\ldots 2\}\int_0^z z^{\frac{1}{2}m}J_{\frac{1}{2}m}(z)\,dz.\Big]$$

34. Show that

$$\int_0^z z^{\frac{1}{2}m}J_{\frac{1}{2}m}(z)\,dz = z^{\frac{1}{2}m}J_{\frac{1}{2}m+1}+z^{\frac{1}{2}m-1}J_{\frac{1}{2}m+2}+\ldots\{1.3.5\ldots(m-1)\}\int_0^z J_m(z)\,dz$$
$$= z^{\frac{1}{2}m}J_{\frac{1}{2}m+1}+z^{\frac{1}{2}m-1}J_{\frac{1}{2}m+2}+\ldots 2(1.3.5\ldots)\{J_{m+1}+J_{m+3}+\ldots\};$$
$$m > -1.$$

35. Prove [8] that

$$\int_0^1 \left\{\frac{z^2}{2!}f_2-\frac{z^4}{4!}f_4+\frac{z^6}{6!}f_6-\ldots\right\}b\,db = \tfrac{1}{2}\left[1-\frac{J_1(2z)}{z}\right],$$

where $f_2 = 1$ $\qquad f_{2r} = {}_2F_1[-(r-1), -r; 1; b^2]$

$f_4 = 1+2b^2$

$f_6 = 1+6b^2+3b^4$

$f_8 = 1+12b^2+18b^4+4b^6$, and z is constant during integration.

This integral occurs in determining the acoustical pressure on a rigid disk vibrating in an infinite plane.

36. Plot the following to obtain polar diagrams [4] of sound distribution from $\phi = -\frac{1}{2}\pi$ to $\frac{1}{2}\pi$:

$$(a)\ p = A_1\left[\frac{J_1(ka\sin\phi)}{ka\sin\phi}\right];$$

$$(b)\ p = A_1\left[\frac{J_1(ka\sin\phi)}{ka\sin\phi}-\frac{2J_2(ka\sin\phi)}{(ka\sin\phi)^2}\right];$$

$$(c)\ p = A_1\left[\frac{J_3(ka\sin\phi)}{(ka\sin\phi)^3}-\frac{6J_4(ka\sin\phi)}{(ka\sin\phi)^4}\right].$$

Take $A_1 = 1$, $a = 10$ cm. and plot two polar curves in each case, (1) $k = 0{\cdot}04$, (2) $k = 0{\cdot}4$. The axis $\phi = 0$ can be taken vertically or horizontally and the curves drawn symmetrically about it. (See Fig. 10.)

37. Show that $\int_0^z \frac{J_n(z)}{z}\,dz = \frac{1}{n}\left\{J_n(z)+2\sum_{r=0}^{\infty}J_{2r+n+2}(z)\right\};\quad n > 0.$

[Use a recurrence formula.]

38. Show that $\displaystyle\int\limits_{1/t}^{\infty} \frac{J_n(1/t)}{t}\,dt = \frac{1}{n}\left\{2\sum_{r=0}^{\infty} J_{2r+n}(1/t) - J_n(1/t)\right\}$; $n > 0$.

[Use a recurrence formula.]

39. Show that $\int\limits_0^{\infty} e^{-\beta z} J_0(\alpha z)\,dz = 1/\sqrt{(\alpha^2+\beta^2)}$ without expanding the integrand into series. [Substitute for $J_0(z)$ from (20) and change the order of integration, i.e. integrate first from 0 to ∞ with respect to z, and second from 0 to π with respect to θ.]

40. Establish formula (32) in the text by aid of formula (21), Chap. II.

41. Evaluate the following integral which occurs in a problem on the symmetrical vibrational modes of a stretched annular membrane of radii a and b:

$$\int\limits_b^a x \log\frac{a}{x}\, J_0(kx)\,dx.$$

$$\left[\frac{1}{k^2}\{J_0(kb) - J_0(ka)\} - \frac{b}{k}\log\frac{a}{b}\, J_1(kb).\right]$$

42. Evaluate the following integral which arises in a problem associated with a loaded submarine cable (see § 5, Chap. VII)

$$-\int\limits^{x} x^{(1+\alpha+2\beta)/2} J_{(1+\alpha)/(\alpha+\beta+2)}[mx^{(\alpha+\beta+2)/2}]\,dx.$$

$$\left[\frac{2}{m(\alpha+\beta+2)} x^{(1+\beta)/2} J_{-(1+\beta)/(\alpha+\beta+2)}[mx^{(\alpha+\beta+2)/2}].\right]$$

IV

THE HYPERGEOMETRIC, GAMMA, AND STRUVE FUNCTIONS; ASYMPTOTIC EXPANSIONS; LOUD SPEAKER HORNS

1. Hypergeometric function of K. F. Gauss

BEFORE dealing with the further development of Bessel functions, we shall introduce two additional functions which are required in subsequent work. The hypergeometric function is one solution of Gauss's equation

$$z(1-z)\frac{d^2y}{dz^2}+\{\gamma-z(\alpha+\beta+1)\}\frac{dy}{dz}-\alpha\beta y = 0,$$

and it is denoted by the letter F. It includes many well-known expansions as particular cases, some of which are given in example 2 at the end of this chapter. By substituting a power series for y in the above equation, we obtain as one solution the hypergeometric function,

$$F(\alpha,\beta,\gamma,z) = \left\{1+\frac{\alpha\beta z}{1!\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)z^2}{2!\gamma(\gamma+1)}+\ldots\right\}. \tag{1}$$

It is absolutely convergent if $|z| < 1$. If $|z| = 1$, it is absolutely convergent provided $R(\gamma-\alpha-\beta) > 0$. The symbolism on the left-hand side of (1) is merely the mathematical shorthand for the function, just as $J_n(z)$ stands for the series which represents Bessel's function of the first kind of order n. In operations involving the hypergeometric function, we shall use the series itself. Suppose we have to evaluate the integral $\int\limits_0^1 F(\alpha,\beta,1,z^2)z\,dz$. Putting $\gamma = 1$ in (1), we obtain

$$\begin{aligned}\int\limits_0^1 F(\alpha,\beta,1,z^2)z\,dz &= \int\limits_0^1 \left\{z+\alpha\beta z^3+\frac{\alpha(\alpha+1)\beta(\beta+1)z^5}{2!2!}+\ldots\right\}dz\\ &= \left[\tfrac{1}{2}z^2+\tfrac{1}{4}\alpha\beta z^4+\frac{\alpha(\alpha+1)\beta(\beta+1)z^6}{(2!)^2 6}+\ldots\right]_0^1\\ &= \tfrac{1}{2}\left\{1+\frac{\alpha\beta}{1!2}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!2.3}+\ldots\right\},\end{aligned}$$

so $$\int\limits_0^1 F(\alpha,\beta,1,z^2)z\,dz = \tfrac{1}{2}F(\alpha,\beta,2,1). \tag{2}$$

Now $F(\alpha,\beta,\gamma,1)$ is convergent only when the real part of

$$(\gamma-\alpha-\beta) > 0,$$

so (2) is valid provided $R(2-\alpha-\beta) > 0$.

In like manner it can be shown that

$$\int\limits_0^a F\left(\alpha,\beta,n,\frac{z^2}{a^2}\right)z^{2n-1}\,dz = \frac{a^{2n}}{2n}F(\alpha,\beta,n+1,1), \tag{3}$$

provided $R(n+1-\alpha-\beta) > 0$, to secure convergence. Also

$$\int\limits_0^a F\left(\alpha,\beta,1,\frac{z^2}{a^2}\right)z^3\,dz = \tfrac{1}{2}a^4\{F(\alpha,\beta,2,1)-\tfrac{1}{2}F(\alpha,\beta,3,1)\}. \tag{4}$$

Instead of integrating the hypergeometric function, suppose we differentiate it, then from (1)

$$\begin{aligned}\frac{\partial}{\partial z}F(\alpha,\beta,\gamma,z) &= \frac{\alpha\beta}{\gamma}+\frac{2\alpha(\alpha+1)\beta(\beta+1)z}{2!\,\gamma(\gamma+1)}+\\ &\qquad +\frac{3\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)z^2}{3!\,\gamma(\gamma+1)(\gamma+2)}+\dots\\ &= \frac{\alpha\beta}{\gamma}\left\{1+\frac{(\alpha+1)(\beta+1)z}{\gamma+1}+\frac{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)z^2}{2!\,(\gamma+1)(\gamma+2)}+\dots\right\}\\ &= \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,z).\end{aligned} \tag{5}$$

More generally, we have

$$\frac{\partial}{\partial z}F(\alpha,\beta,\gamma,\chi) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,\chi)\frac{\partial\chi}{\partial z}. \tag{6}$$

2. The Gamma function $\Gamma(z)$

This function [84] was introduced by Euler. It is represented by the Greek capital letter gamma. The function is the limiting value of a product when the number of terms therein tends to infinity. The reciprocal of this function is defined to be

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{m=1}^{\infty}\left\{\left(1+\frac{z}{m}\right)e^{-z/m}\right\}, \tag{7}$$

where capital π signifies that the *product* of the terms is to be taken, and γ is a well-known constant due to Euler, which is associated with his name. Numerically

$$\gamma = \lim_{m\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m}-\log m\right] \doteqdot 0{\cdot}5772. \tag{8}$$

Taking the product in (7) we have

$$ze^{\gamma z}\lim_{m\to\infty}\left\{(1+z)e^{-z}(1+\tfrac{1}{2}z)e^{-\frac{1}{2}z}...\left(1+\frac{z}{m}\right)e^{-z/m}\right\}$$

$$= ze^{\gamma z}\lim_{m\to\infty}\frac{(1+z)(2+z)...(m+z)}{m!}e^{-z(1+\frac{1}{2}+...1/m)}. \tag{9}$$

Now $e^{z\log m} = e^{\log m^z} = m^z$. Multiplying (9) above and below by $e^{z\log m}$ we get

$$ze^{\gamma z}\lim_{m\to\infty}\frac{(z+1)(z+2)...(z+m)}{m!m^z}e^{-z(1+\frac{1}{2}+\frac{1}{3}+...1/m-\log m)};$$

so
$$\frac{1}{\Gamma(z)} = \lim_{m\to\infty}\frac{z(z+1)(z+2)...(z+m)}{m!m^z}e^{-\gamma z}e^{\gamma z}$$

or
$$\Gamma(z) = \lim_{m\to\infty}\frac{m!m^z}{z(z+1)(z+2)...(z+m)}. \tag{10}$$

It can also be shown that

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt, \tag{10a}$$

provided the real part of z is positive. Writing $(z+1)$ in place of z in (10) we get

$$\Gamma(1+z) = \lim_{m\to\infty}\frac{m!m^{1+z}}{(z+1)(z+2)...(z+m+1)} = z\lim_{m\to\infty}\frac{m!m^z m}{z(z+1)...(z+m+1)}$$

$$= z\Gamma(z)\lim_{m\to\infty}\frac{m}{z+m+1}. \tag{10b}$$

Thus
$$\Gamma(1+z) = z\Gamma(z) = z(z-1)(z-2)\Gamma(z-2). \tag{11}$$

Consequently $\Gamma(1+z)$ is sometimes called the *factorial* function† and it is written $z!$. From (11) it follows that when n is a positive integer

$$\Gamma(1+n) = n(n-1)(n-2)...1 = n! \tag{11a}$$

It can also be shown that

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z. \tag{12}$$

If $z = \frac{1}{2}$ we have from (12)

$$\{\Gamma(\tfrac{1}{2})\}^2 = \pi, \quad \text{so} \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}; \tag{13}$$

the positive root being chosen since $\Gamma(\frac{1}{2}) > 0$. Again, from (12) $\dfrac{1}{\Gamma(1-z)} = \dfrac{\Gamma(z)\sin\pi z}{\pi}$, and it is clear that $\dfrac{1}{\Gamma(1-z)} = 0$, or $\Gamma(1-z) = \infty$ when z is a positive non-zero integer. Alternatively, $\dfrac{1}{\Gamma(1+z)} = 0$ or

† In this book we shall reserve $z!$ to apply to positive integral values of z.

$\Gamma(1+z) = \infty$, when z is a negative integer, a result of importance in establishing the relationship $J_{-n}(z) = (-1)^n J_n(z)$ in § 4. The function $\Gamma(1+z)$ is shown graphically in Fig. 11. It is positive for all values of $z > -1$, whilst when z is a negative integer it is infinite. Between successive negative integral values it is alternately positive and negative. To evaluate $\Gamma(z)$ when z is half an odd negative integer

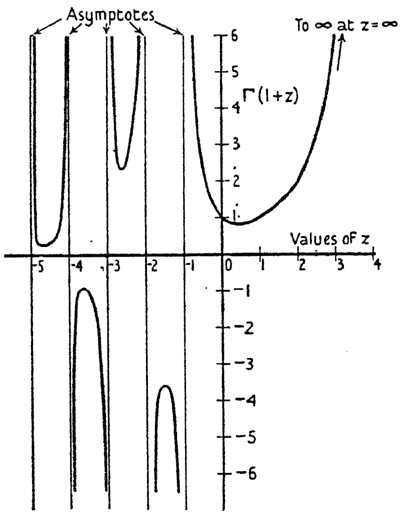


FIG. 11. The gamma function $\Gamma(1+z)$.

we proceed as follows:

$$\Gamma(z+m) = (z+m-1)(z+m-2)...(z+1)z\Gamma(z),$$

so
$$\Gamma(z) = \Gamma(z+m)/(z+m-1)...z. \tag{14}$$

Putting $z+m = \frac{1}{2}$, or $m = \frac{1}{2}-z$, this being a positive integer,

$$\Gamma(z) = \sqrt{\pi}/(-\tfrac{1}{2})(-\tfrac{3}{2})...z. \tag{14a}$$

There is an important formula due to Gauss by means of which the hypergeometric series can be evaluated when the argument is unity and α, β, γ have numerical values. It is [84, 77]

$$F(\alpha,\beta,\gamma,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{15}$$

provided the real parts of γ and $(\gamma-\alpha-\beta)$ exceed zero, i.e. $R(\gamma) > 0$, $R(\gamma-\alpha-\beta) > 0$.

3. Example

Evaluate $F(-\frac{3}{2}, \frac{1}{2}, 3, 1)$. Using (15) we get

$$F(-\tfrac{3}{2}, \tfrac{1}{2}, 3, 1) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{9}{2})\Gamma(\frac{5}{2})},$$

and by aid of (11) we obtain

$$F(-\tfrac{3}{2}, \tfrac{1}{2}, 3, 1) = 2.1\times 3.2.1/\tfrac{7}{2}.\tfrac{5}{2}.\tfrac{3}{2}.\tfrac{1}{2}.\sqrt{\pi}\times\tfrac{3}{2}.\tfrac{1}{2}.\sqrt{\pi}, \quad \text{since } \Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$
$$= 2^8/105\pi.$$

This represents the sum of the hypergeometric series, namely,

$$F(-\tfrac{3}{2}, \tfrac{1}{2}, 3, 1) = 1+\frac{(-\frac{3}{2})\frac{1}{2}}{1!3}+\frac{(-\frac{3}{2})(-\frac{1}{2})(\frac{1}{2})(\frac{3}{2})}{2!3.4}+\dots.$$

Thus $$\frac{2^8}{105\pi} = 1-\tfrac{1}{4}+\frac{3}{128}+\dots \doteqdot 0{\cdot}7734+\text{remainder},$$

or $$0{\cdot}775 = 0{\cdot}7734+R_n.$$

4. Series for $J_\nu(z)$ when ν is unrestricted

When the order ν is unrestricted, i.e. it is either integral, fractional, or complex, a Bessel function† of the first kind of order ν is defined to be a solution of the differential equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{\nu^2}{z^2}\right)y = 0,$$

this being identical with (11), Chap. I, when ν is a positive integer. The first solution of this equation is given by

$$a_0 z^\nu\left\{1-\frac{(\frac{1}{2}z)^2}{1!(\nu+1)}+\frac{(\frac{1}{2}z)^4}{2!(\nu+1)(\nu+2)}+\dots\right\}.$$

If we make $a_0 = \dfrac{1}{2^\nu\Gamma(\nu+1)}$, we obtain

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\left\{1-\frac{(\frac{1}{2}z)^2}{(\nu+1)}+\frac{(\frac{1}{2}z)^4}{2!(\nu+1)(\nu+2)}-\dots\right\} \tag{16}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2}z)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)}. \tag{16a}$$

This agrees with the definition in Chapter II for integral values of ν, since $\Gamma(\nu+r+1)$ is then $\Gamma(n+r+1) = (n+r)!$, by (11a). For this reason the above value of a_0 is chosen.

We are now in a position to prove (4)‡ in Chap. II, namely, that

† When ν is not integral $J_\nu(z)$ cannot be regarded as a coefficient as in Chap. I, § 3.
‡ See also examples 24 and 25 at the end of Chapter III.

$J_{-n}(z) = (-1)^n J_n(z)$. Writing $-n$ for ν in (16 a) we get

$$J_{-n}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{2r-n}}{r!\,\Gamma(1+r-n)}.$$

From § 2, $1/\Gamma(1+r-n)$ is zero when $(r-n)$ is a negative integer, a result which holds for all values of r from 0 to $(n-1)$. Thus we are only concerned with the series of values $r = (n+m)$ for $m = 0$ to ∞. Writing $r = (n+m)$ in the above series, we obtain

$$J_{-n}(z) = \sum_{m=0}^{\infty} \frac{(-1)^{n+m}(\frac{1}{2}z)^{n+2m}}{m!\,(n+m)!} \tag{17}$$

since $\Gamma(1+m) = m!$ and the summation applies to the terms obtained by putting $m = 0, 1, 2, \ldots$. Thus comparing (17) with (3), Chap. II, it is seen that

$$J_{-n}(z) = (-1)^n J_n(z). \tag{17 a}$$

It can be shown that $J_\nu(z)$ obeys the recurrence and other relationships established in Chapter II.

5. Complex variable

When z is complex we have $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, and when ν is not integral z^ν is many-valued since $z^\nu = r^\nu e^{i\nu(\theta+2n\pi)}$. To avoid ambiguity we take $z^\nu = r^\nu e^{i\nu\theta}$, θ being chosen so that $-\pi < \theta \leqslant \pi$, which means that the angle $-\pi$ is not used† (see Fig. 12 A). Suppose $z = (x-iy)$, where x and y are positive real quantities, then the position of r is in the fourth quadrant in Fig. 12 A. Let it be required to find the value of a function, say $\phi(-z)$, when the argument $-z$ is in the second quadrant. From Fig. 12 A it is seen that the vector is rotated $+\pi$, so if $z = re^{i\theta}$, we have $(-z) = re^{i(\theta+\pi)} = ze^{i\pi}$. Thus

$$\phi(-z) = \phi(ze^{i\pi}).$$

Applying this to the case of $J_\nu(z)$, we have

$$\begin{aligned} J_\nu(-z) = J_\nu(ze^{i\pi}) &= \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}ze^{i\pi})^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \\ &= e^{i\nu\pi} \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}, \quad \text{since } e^{2\pi mi} = 1, \\ &= e^{i\nu\pi}[J_\nu(z)], \end{aligned} \tag{18}$$

the square brackets signifying that the value of the function within

† It may be convenient in certain cases to alter the angle range, i.e. to depart from the principal branch of the function.

is known. The same result is obtained if we commence with z in the third quadrant. Since θ must not exceed π, it follows that when $-\pi < \theta \leqslant 0$, $J_\nu(-z) = e^{i\nu\pi}[J_\nu(z)]$. Starting in the first or second quadrants, if we know $J_\nu(z)$ and want to find $J_\nu(-z)$, we have $z = re^{i\theta}$ and $(-z) = re^{i(\theta-\pi)}$, the vector now being rotated $-\pi$ as

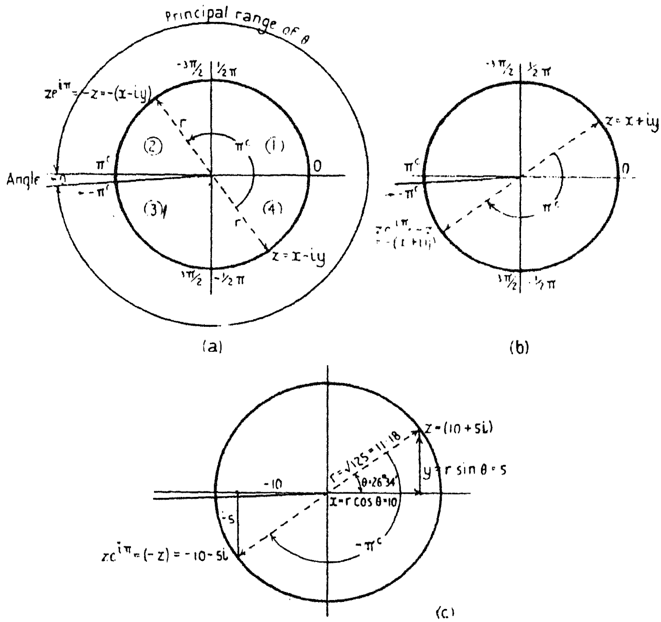


FIG. 12. Illustrating complex variables: (a) in changing from quadrants 3, 4 to 1, 2 the multiplier is $e^{i\pi}$; (b) in changing from 1, 2 to 3, 4 the multiplier is $e^{-i\pi}$. The line or barrier at $-\pi$ must not be crossed in passing from one quadrant to another. Hence we cannot go clockwise from 4 to 2, or anti-clockwise from 2 to 4. (c) Diagram illustrating § 12.

shown in Fig. 12B. Thus when $0 < \theta \leqslant \pi$, $(-z) = ze^{-i\pi}$ and

$$J_\nu(-z) = e^{-i\nu\pi}[J_\nu(z)] \tag{18a}$$

The above results can be summarized as follows:

When z is in the third or fourth quadrants,

$$-\pi < \theta \leqslant 0,\ (-z) = ze^{i\pi} \quad \text{and} \quad J_\nu(-z) = e^{i\nu\pi}[J_\nu(z)].$$

When z is in the first or second quadrants,

$$0 < \theta \leqslant \pi,\ (-z) = ze^{-i\pi} \quad \text{and} \quad J_\nu(-z) = e^{-i\nu\pi}[J_\nu(z)].$$

When $\theta = 0$, $z = r$ is *positive* and real so $J_\nu(-z) = e^{i\nu\pi}[J_\nu(z)]$, but

if for convenience z is taken to be negative, we have

$$J_\nu(z) = e^{i\nu\pi}J_\nu(-z), \tag{18b}$$

where $(-z)$ is positive. As a case in point, suppose we know $J_{\frac{1}{2}}(\frac{1}{2}\pi)$ and we want to find $J_{\frac{1}{2}}(-\frac{1}{2}\pi)$. Then

$$J_{\frac{1}{2}}(-\tfrac{1}{2}\pi) = e^{\frac{1}{2}\pi i}[J_{\frac{1}{2}}(\tfrac{1}{2}\pi)] = i[\sqrt{(4/\pi^2)}\sin\tfrac{1}{2}\pi] = 2i/\pi.$$

Formulae for the functions of the second and third kinds are obtained as shown below.

$$\begin{aligned} Y_\nu(-z) &= \frac{\cos\nu\pi J_\nu(-z)-J_{-\nu}(-z)}{\sin\nu\pi} \\ &= \frac{e^{i\nu\pi}\cos\nu\pi\, J_\nu(z)-e^{-i\nu\pi}\cos\nu\pi\, J_\nu(z)}{\sin\nu\pi}+\frac{e^{-i\nu\pi}\cos\nu\pi\, J_\nu(z)-e^{-i\nu\pi}\, J_{-\nu}(z)}{\sin\nu\pi} \\ &= e^{-i\nu\pi}Y_\nu(z)+2i\cos\nu\pi\, J_\nu(z), \end{aligned} \tag{19}$$

$+z$ being in quadrants 3 or 4 where $-\pi < \theta \leqslant 0$.

$$\begin{aligned} Y_\nu(-z) &= \frac{e^{-i\nu\pi}\cos\nu\pi\, J_\nu(z)-e^{i\nu\pi}J_{-\nu}(z)}{\sin\nu\pi} \\ &= \frac{e^{i\nu\pi}}{\sin\nu\pi}\{\cos\nu\pi\, J_\nu(z)-J_{-\nu}(z)\}-\frac{\cos\nu\pi\, J_\nu(z)}{\sin\nu\pi}(e^{+i\nu\pi}-e^{-i\nu\pi}) \\ &= e^{i\nu\pi}Y_\nu(z)-2i\cos\nu\pi\, J_\nu(z), \end{aligned} \tag{19a}$$

$+z$ being in quadrants 1 or 2 where $0 < \theta \leqslant \pi$.

$$\begin{aligned} H_\nu^{(1)}(-z) &= J_\nu(-z)+iY_\nu(-z) \\ &= e^{i\nu\pi}J_\nu(z)-(e^{i\nu\pi}+e^{-i\nu\pi})J_\nu(z)+ie^{-i\nu\pi}Y_\nu(z) \\ &= -e^{-i\nu\pi}H_\nu^{(2)}(z) = e^{-i\nu\pi}H_\nu^{(1)}(z)-2e^{-i\nu\pi}J_\nu(z), \end{aligned} \tag{20}$$

$+z$ being in quadrants 3 or 4 where $-\pi < \theta \leqslant 0$.

$$\begin{aligned} H_\nu^{(1)}(-z) &= e^{-i\nu\pi}J_\nu(z)+(e^{i\nu\pi}+e^{-i\nu\pi})J_\nu(z)+ie^{i\nu\pi}Y_\nu(z) \\ &= e^{i\nu\pi}H_\nu^{(1)}(z)+2e^{-i\nu\pi}J_\nu(z), \end{aligned} \tag{20a}$$

$+z$ being in quadrants 1 or 2 where $0 < \theta \leqslant \pi$. Similarly

$$H_\nu^{(2)}(-z) = 2\cos\nu\pi\, H_\nu^{(2)}(z)+e^{i\nu\pi}H_\nu^{(1)}(z), \tag{21}$$

$+z$ being in quadrants 3 or 4 where $-\pi < \theta \leqslant 0$.

$$H_\nu^{(2)}(-z) = -e^{i\nu\pi}H_\nu^{(1)}(z) = e^{i\nu\pi}H_\nu^{(2)}(z)-2e^{i\nu\pi}J_\nu(z), \tag{21a}$$

$+z$ being in quadrants 1 or 2 where $0 < \theta \leqslant \pi$. An example illustrating the use of these formulae is given in § 12.

6. Relationship between Bessel and circular functions when ν is half an odd integer

Having introduced the gamma function, we are now able to consider Bessel functions with fractional indices. In (16) put $\nu = \frac{1}{2}$, then

$$J_{\frac{1}{2}}(z) = \frac{(\frac{1}{2}z)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\left\{1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-\ldots\right\}. \tag{22}$$

But $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$, so we get on multiplying (22) above and below by z

$$J_{\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\left\{z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots\right\}, \tag{22 a}$$

or

$$J_{\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\sin z = Y_{-\frac{1}{2}}(z) \tag{23}$$

from (10), Chap. II. In like manner it can be shown that

$$J_{-\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\cos z = -Y_{\frac{1}{2}}(z) \tag{24}$$

from (10), Chap. II. Thus

$$\sqrt{(\tfrac{1}{2}\pi z)}\{J_{-\frac{1}{2}}(z)\pm iJ_{\frac{1}{2}}(z)\} = \cos z\pm i\sin z = e^{\pm iz}. \tag{25}$$

Also

$$J_{\frac{3}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\frac{\sin z}{z}-\cos z\right\}, \tag{26}$$

$$J_{-\frac{3}{2}}(z) = -\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\sin z+\frac{\cos z}{z}\right\}. \tag{27}$$

If in (23) and (24) we replace z by iz

$$J_{\frac{1}{2}}(zi) = \sqrt{\left(\frac{2}{\pi zi}\right)}\sin(zi) = \sqrt{\left(\frac{2i}{\pi z}\right)}\sinh z = (1+i)\sqrt{\left(\frac{1}{\pi z}\right)}\sinh z, \tag{28}$$

$$J_{-\frac{1}{2}}(zi) = \sqrt{\left(\frac{2}{\pi zi}\right)}\cos(zi) = \sqrt{\left(\frac{2}{\pi zi}\right)}\cosh z = (1-i)\sqrt{\left(\frac{1}{\pi z}\right)}\cosh z, \tag{28 a}$$

since, from Fig. 18, $i^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(1+i)$ and $i^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}(1-i)$.

In general, when ν is half an odd positive integer, Bessel functions can be expressed in terms of circular functions [93]. Thus, when $n \geqslant 0$,

$$J_{n+\frac{1}{2}}(z) = \frac{2(\frac{1}{2}z)^{n+\frac{1}{2}}}{\sqrt{\pi}n!}\left\{\left(1+\frac{d^2}{dz^2}\right)^n\frac{\sin z}{z}\right\}. \tag{29}$$

Also

$$Y_{n+\frac{1}{2}}(z) = -\frac{2(\frac{1}{2}z)^{n+\frac{1}{2}}}{\sqrt{\pi}n!}\left\{\left(1+\frac{d^2}{dz^2}\right)^n\frac{\cos z}{z}\right\}. \tag{29a}$$

7. Formulae for $J_\nu(z)$ using the gamma function

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^\pi e^{\pm iz\cos\theta}\sin^{2\nu}\theta\, d\theta \tag{30}$$

$$= \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\, d\theta \tag{30a}$$

$$= \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^{\frac{1}{2}\pi} \cos(z\cos\theta)\sin^{2\nu}\theta\, d\theta \tag{30b}$$

$$= \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^{\frac{1}{2}\pi} \cos(z\sin\theta)\cos^{2\nu}\theta\, d\theta \tag{30c}$$

$$= \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^1 (1-t^2)^{\nu-\frac{1}{2}}\cos zt\, dt \tag{30d}$$

$$= \frac{2(\frac{1}{2}z)^{-\nu}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\nu)} \int\limits_1^\infty \frac{\sin zt}{(t^2-1)^{\nu+\frac{1}{2}}}\, dt \tag{30e}$$

$$= \frac{(2\nu-1)(\frac{1}{2}z)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^{\frac{1}{2}\pi} \sin(z\cos\theta)\sin^{2\nu-2}\theta\cos\theta\, d\theta. \tag{30f}$$

Formulae (30) to (30 d) inclusive are valid when the real part of ν exceeds $-\frac{1}{2}$, i.e. $R(\nu) > -\frac{1}{2}$. (30 e) is valid when $z > 0$, $-\frac{1}{2} < R(\nu) < \frac{1}{2}$. (30 f), which is obtained from (30 b) by partial integration, is valid when $R(\nu) > \frac{1}{2}$. The results (30) to (30 c) can be established by expanding the integrands and integrating term by term using the formulae

$$\int\limits_0^{\frac{1}{2}\pi} \sin^{2\nu}z\, dz = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}; \quad \int\limits_0^{\frac{1}{2}\pi} \sin^{2\nu}z\cos^{2\mu}z\, dz = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\frac{1}{2})}{2\Gamma(\nu+\mu+1)}.$$

8. Struve functions

The simplest way of introducing these functions, which are named after the German mathematician H. Struve, is by means of an example. Suppose we evaluate

$$\int\limits_0^{\frac{1}{2}\pi} e^{iz\sin\theta}\, d\theta = \int\limits_0^{\frac{1}{2}\pi} \{\cos(z\sin\theta)+i\sin(z\sin\theta)\}\, d\theta.$$

From (26), Chap. III, the integral of the real part is

$$\int_0^{\frac{1}{2}\pi} \cos(z\sin\theta)\,d\theta = \tfrac{1}{2}\pi J_0(z).$$

To evaluate the imaginary part we have

$$\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi} \left\{z\sin\theta - \frac{z^3}{3!}\sin^3\theta + \frac{z^5}{5!}\sin^5\theta - \ldots\right\} d\theta.$$

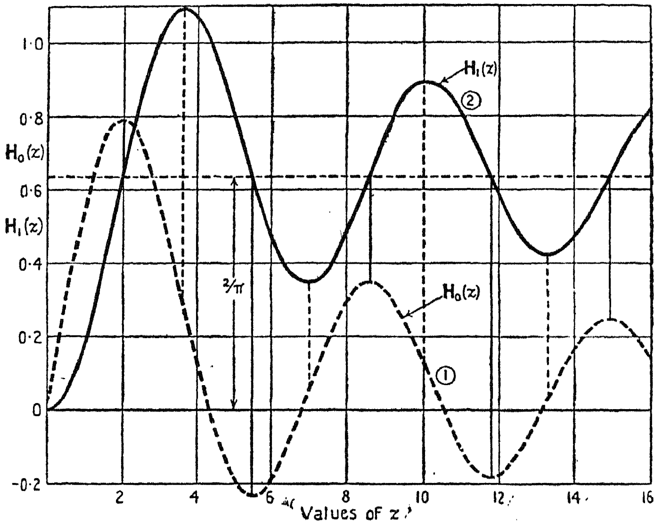


FIG. 13. Struve's functions $\mathbf{H}_0(z)$ and $\mathbf{H}_1(z)$. The zeros of $\mathbf{H}_0(z)$ are 4·333, 6·782, 10·47, etc. When $z \to \infty$, $\mathbf{H}_1(z) \to 2/\pi$.

When m is a positive integer $\int_0^{\frac{1}{2}\pi} \sin^{2m+1}\theta\,d\theta = \dfrac{2^m m!}{1.3.5\ldots(2m+1)}$, so that the imaginary part of the original integral is

$$\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\,d\theta = \frac{z}{1^2} - \frac{z^3}{1^2.3^2} + \frac{z^5}{1^2.3^2.5^2} - \ldots \tag{31}$$

$$= \tfrac{1}{2}\pi\mathbf{H}_0(z), \tag{32}$$

where
$$\mathbf{H}_0(z) = \frac{2}{\pi}\left\{\frac{z}{1^2} - \frac{z^3}{1^2.3^2} + \frac{z^5}{1^2.3^2.5^2} - \ldots\right\} \tag{33}$$

(see Fig. 13, curve 1) is defined to be Struve's function of order zero. It is used in acoustical problems [8, 17] associated with the determination of the fluid pressure on a vibrating surface.

Since $\sin\theta$ from 0 to $\frac{1}{2}\pi$ and $\cos\theta$ from $\frac{1}{2}\pi$ to 0 are identical, by regarding the areas formed by integration, it is seen that

$$\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi} \sin(z\cos\theta)\,d\theta = \tfrac{1}{2}\pi\mathbf{H}_0(z).$$

The Struve function of order ν is defined to be [93]

$$\mathbf{H}_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \sin(z\cos\theta)\sin^{2\nu}\theta\,d\theta, \tag{34}$$

provided $R(\nu) > -\frac{1}{2}$. For any value of ν

$$\mathbf{H}_\nu(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})}. \tag{35}$$

It is of interest to compare (34) and (35) with the corresponding formulae for the Bessel function of the first kind. Thus,

$$J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta, \tag{36}$$

provided $R(\nu) > -\frac{1}{2}$. For any value of ν

$$J_\nu(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{\nu+2r}}{\Gamma(r+1)\Gamma(\nu+r+1)}. \tag{37}$$

Since $e^{iz\cos\theta} = \cos(z\cos\theta)+i\sin(z\cos\theta)$, it follows from (34) and (36) that

$$J_\nu(z)+i\mathbf{H}_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} e^{iz\cos\theta}\sin^{2\nu}\theta\,d\theta, \tag{38}$$

provided $R(\nu) > -\frac{1}{2}$. By writing $\theta = (\frac{1}{2}\pi-\phi)$ it is easy to show that

$$J_\nu(z)+i\mathbf{H}_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} e^{iz\sin\theta}\cos^{2\nu}\theta\,d\theta, \tag{39}$$

provided $R(\nu) > -\frac{1}{2}$.

The Struve function of unit order is of considerable importance in the theory of loud-speaker diaphragms [83]. Using (35) and writing $\nu = 1$, $r = 0, 1, 2,\ldots$ we find that

$$\mathbf{H}_1(z) = \frac{z^2}{2^2}\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}-\frac{z^4}{2^4}\frac{1}{\Gamma(\frac{5}{2})\Gamma(\frac{7}{2})}+\frac{z^6}{2^6}\frac{1}{\Gamma(\frac{7}{2})\Gamma(\frac{9}{2})}-\ldots. \tag{40}$$

Now $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$; $\Gamma(\frac{5}{2}) = \frac{3}{2}\frac{1}{2}\sqrt{\pi}$, and so on. Thus (40) can be written

$$\mathbf{H}_1(z) = \frac{2}{\pi}\left\{\frac{z^2}{1^2.3} - \frac{z^4}{1^2.3^2.5} + \frac{z^6}{1^2.3^2.5^2.7} - \ldots\right\}. \tag{41}$$

$\mathbf{H}_1(z)$ is plotted in Fig. 13, curve 2.

Using (41) it is easy to show that

$$z\mathbf{H}_1(z) = \int^z z\mathbf{H}_0(z)\,dz, \quad \text{or} \quad \frac{d}{dz}\{z\mathbf{H}_1(z)\} = z\mathbf{H}_0(z). \tag{42}$$

By differentiating the series in (33) we get

$$\frac{d}{dz}\{\mathbf{H}_0(z)\} = \frac{2}{\pi} - \mathbf{H}_1(z), \quad \text{or} \quad \mathbf{H}_0(z) = \frac{2z}{\pi} - \int^z \mathbf{H}_1(z)\,dz. \tag{42 a}$$

9. Recurrence formulae for $\mathbf{H}_\nu(z)$

As with Bessel functions, recurrence formulae can also be established for Struve functions [93]. It can be shown that

$$\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}_\nu'(z) - \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}, \tag{43}$$

$$\mathbf{H}_{\nu+1}(z) + \mathbf{H}_{\nu-1}(z) = \frac{2\nu}{z}\mathbf{H}_\nu(z) + \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}, \tag{44}$$

which should be compared with (22) and (23), Chap. II. It can also be shown that†

$$\frac{d}{dz}\{z^\nu\mathbf{H}_\nu(z)\} = z^\nu\mathbf{H}_{\nu-1}(z), \quad \text{or} \quad z^\nu\mathbf{H}_\nu(z) = \int^z z^\nu\mathbf{H}_{\nu-1}(z)\,dz, \tag{45}$$

$$\frac{d}{dz}\{z^{-\nu}\mathbf{H}_\nu(z)\} = \frac{1}{2^\nu\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} - z^{-\nu}\mathbf{H}_{\nu+1}(z), \tag{46}$$

or

$$z^{-\nu}\mathbf{H}_\nu(z) = \frac{z}{2^\nu\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} - \int^z \frac{\mathbf{H}_{\nu+1}(z)}{z^\nu}\,dz. \tag{47}$$

10. Example

Evaluate

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-iz\cos\theta}\cos\theta\,d\theta.$$

Expanding the integrand we get

$$2\int_0^{\frac{1}{2}\pi} \{\cos(z\cos\theta) - i\sin(z\cos\theta)\}\,d\sin\theta$$

$$= 2\int_0^{\frac{1}{2}\pi} \cos(z\cos\theta)\,d\sin\theta - 2i\int_0^{\frac{1}{2}\pi} \sin(z\cos\theta)\,d\sin\theta. \tag{48}$$

† Additional formulae are given in the list on p. 167.

Taking the first integral in (48) by parts we have

$$2[\sin\theta\cos(z\cos\theta)]_0^{\frac{1}{2}\pi}-2z\int\limits_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\sin^2\theta\,d\theta.$$

Inserting the limits in the first term and using (34) to evaluate the second, we find that

$$2\int\limits_0^{\frac{1}{2}\pi}\cos(z\cos\theta)\,d\sin\theta = 2-\pi\mathbf{H}_1(z). \tag{49}$$

Taking the second integral in (48) by parts, we obtain

$$2i[\sin\theta\sin(z\cos\theta)]_0^{\frac{1}{2}\pi}+2iz\int\limits_0^{\frac{1}{2}\pi}\cos(z\cos\theta)\sin^2\theta\,d\theta.$$

The first term vanishes at the limits of integration, whilst by (36) the value of the integral is $i\pi J_1(z)$, so

$$2i\int\limits_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\,d\sin\theta = i\pi J_1(z). \tag{50}$$

Subtracting (50) from (49) we find that

$$\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}e^{-iz\cos\theta}\cos\theta\,d\theta = 2-\pi\{\mathbf{H}_1(z)+iJ_1(z)\}. \tag{51}$$

11. Asymptotic expansions of $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, and $\mathbf{H}_\nu(z)$

In general the series (16) or (16 a) is convenient for numerical calculation when the convergence is fairly rapid. If z and ν are such that convergence is slow, which usually means that $z \gg 1$, series (16) is not well suited for computation. It is customary for large values of z to use what is known as an asymptotic expansion. When, however, $\nu = n\pm\frac{1}{2}$, $J_\nu(z)$ and $Y_\nu(z)$ can be expressed in terms of circular functions so an asymptotic expansion is not required [see (29), (29 a)]. When z is large, and $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, it can be shown that

$$J_\nu(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\{\zeta_\nu(z)\cos\psi-\xi_\nu(z)\sin\psi\} \tag{52}$$

and

$$Y_\nu(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\{\zeta_\nu(z)\sin\psi+\xi_\nu(z)\cos\psi\}, \tag{53}$$

where

$$\zeta_\nu(z) = 1-\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2}+ \\ +\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)(4\nu^2-7^2)}{4!(8z)^4}-...+R_p, \quad (54)$$

$$\xi_\nu(z) = \frac{(4\nu^2-1^2)}{1!(8z)}-\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{3!(8z)^3}+...+R_s, \quad (55)$$

$$\psi = (z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi);$$

R_p and R_s are the remainders after p and s terms of the respective series have been taken. It should be observed that

$$\zeta_\nu(z) = \zeta_{-\nu}(z) = \zeta_\nu(-z) \quad \text{and} \quad \xi_\nu(z) = \xi_{-\nu}(z) = -\xi_\nu(-z).$$

When $-z$ is negative, we have from formula (18), § 5,

$$J_\nu(-z) = e^{i\nu\pi}J_\nu(+z),$$

and from (19) $\quad Y_\nu(-z) = e^{-i\nu\pi}Y_\nu(+z)+2i\cos\nu\pi J_\nu(+z),$

where $(+z)$ is positive. Thus the values of $J_\nu(z)$ and $Y_\nu(z)$ can be found for large negative values of z by aid of these formulae and (52), (53).

If $\nu^2/z \ll 1$ and $z \gg 1$, each term of the series for $\zeta_\nu(z)$ is at first numerically smaller than its predecessor. After a certain point is reached, however, each term will be numerically larger than its predecessor, so (54) regarded as an infinite series is divergent for all values of z, and a certain term is the smallest. This can be shown approximately in the following way. The ratio of the pth to the $(p-1)$th term is $u_p/u_{p-1} = \dfrac{\{4\nu^2-(4p-7)^2\}\{4\nu^2-(4p-5)^2\}}{(2p-3)(2p-2)(8z)^2}$, and when $p \gg \nu$ we can write

$$u_p/u_{p-1} \doteqdot \frac{(4p-7)^2(4p-5)^2}{(2p-3)(2p-2)(8z)^2} \doteqdot \frac{256(p-1)^4}{256(p-1)^2z^2} = (p-1)^2/z^2.$$

Thus the smallest term occurs *approximately* when

$$(p-1)^2 = z^2 \quad \text{or} \quad p = 1+|z|.$$

If the pth term is the smallest it will be $\ll 1$, and *usually* it is numerically $> R_p$. If terms beyond the smallest are included in a computation, the error steadily increases with increase in the number of terms. Thus by retaining only those terms as far as the smallest, the value of $\zeta_\nu(z)$ can be calculated with an error not exceeding that term. The value of $\xi_\nu(z)$ can be found in the same way. If z is fairly large and ν not too large, $\zeta_\nu(z)$ and $\xi_\nu(z)$ can be calculated to several decimal

places by using only a *few* terms of the series. For instance, when $\nu = 1$ and $z = 10$, the first term of $\xi_\nu(z)$ and the first two of $\zeta_\nu(z)$ gives a result correct to four decimal places. A series of this type is known as an asymptotic expansion.

When z is complex, $J_\nu(z)$ and $Y_\nu(z)$ can be computed from the series for $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, if desired, using the following relationships:

$$J_\nu(z) = \tfrac{1}{2}\{H_\nu^{(1)}(z)+H_\nu^{(2)}(z)\} \tag{56}$$

$$Y_\nu(z) = -\tfrac{1}{2}i\{H_\nu^{(1)}(z)-H_\nu^{(2)}(z)\}. \tag{57}$$

The asymptotic expansions of the functions of the third kind are:

$$H_\nu^{(1)}(z) = J_\nu(z)+iY_\nu(z) = \sqrt{\left(\frac{2}{\pi z}\right)}e^{i(z-\frac{1}{4}\pi-\frac{1}{2}\nu\pi)}\{\zeta_\nu(z)+i\xi_\nu(z)\} \tag{58}$$

$$= \frac{e^{-r\sin\theta}}{\sqrt{(\frac{1}{2}\pi r)}}e^{i\{r\cos\theta-\frac{1}{2}\theta-\frac{1}{4}(2\nu+1)\pi\}}\{\zeta_\nu(z)+i\xi_\nu(z)\}, \tag{59}$$

when $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, i.e. in the first and fourth quadrants where $R(z) \geqslant 0$, and $z = re^{i\theta} = r(\cos\theta+i\sin\theta)$.

$$H_\nu^{(2)}(z) = J_\nu(z)-iY_\nu(z) = \sqrt{\left(\frac{2}{\pi z}\right)}e^{-i(z-\frac{1}{4}\pi-\frac{1}{2}\nu\pi)}\{\zeta_\nu(z)-i\xi_\nu(z)\} \tag{60}$$

$$= \frac{e^{r\sin\theta}}{\sqrt{(\frac{1}{2}\pi r)}}e^{-i\{r\cos\theta+\frac{1}{2}\theta-\frac{1}{4}(2\nu+1)\pi\}}\{\zeta_\nu(z)-i\xi_\nu(z)\}, \tag{61}$$

when $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, i.e. in the first and fourth quadrants where $R(z) \geqslant 0$. If θ lies outside the range $\pm\frac{1}{2}\pi$, it is necessary to use the formulae in § 5 as shown in the example given later on.

The asymptotic expansion of Struve's function [93] is

$$\mathbf{H}_\nu(z) = Y_\nu(z)+\left\{\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{\Gamma(r+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-r)(\frac{1}{2}z)^{2r-\nu+1}}+S_n\right\}, \tag{62}$$

provided $R(n+\frac{1}{2}-\nu) \geqslant 0$, and the appropriate value of $Y_\nu(z)$ from above is used.

By aid of the foregoing formulae, it is easy to find the limiting values of the various functions when $r \to \infty$. Thus from (59) and (61)

$$H_\nu^{(1)}(z) \to 0;\ H_\nu^{(2)}(z) \to \infty \quad \text{when } 0 < \theta \leqslant \tfrac{1}{2}\pi, \tag{63}$$

$$H_\nu^{(1)}(z) \to \infty;\ H_\nu^{(2)}(z) \to 0 \quad \text{when } -\tfrac{1}{2}\pi \leqslant \theta < 0, \tag{64}$$

$$H_\nu^{(1)}(z) \to 0;\ H_\nu^{(2)}(z) \to 0 \quad \text{when } \theta = 0, \text{ i.e. } z \text{ is real and positive.} \tag{65}$$

From (56), (57), (63), (64), and (65)

$$J_\nu(z) \to \infty;\ Y_\nu(z) \to \infty \quad \text{when } z \text{ is complex and } -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi,\ \theta \neq 0. \tag{66}$$

$$J_\nu(z) \to 0;\ Y_\nu(z) \to 0 \quad \text{when } \theta = 0, \text{ i.e. } z \text{ is real and positive.} \tag{67}$$

12. Example

Evaluate $J_{\frac{1}{3}}(10+5i)$, $Y_{\frac{1}{3}}(10+5i)$, $J_{\frac{1}{3}}(-10-5i)$, $Y_{\frac{1}{3}}(-10-5i)$. From Fig. 12c and formula (59) we have $r = \sqrt{125} = 11{\cdot}18$, $r\sin\theta = 5$, $r\cos\theta = 10^c = 573^\circ$, $\theta = 26^\circ\,34'$, $\frac{1}{2}\theta = 13^\circ\,17'$, $\frac{1}{4}(2\nu+1)\pi = 75^\circ$, $r\cos\theta-\frac{1}{2}\theta-\frac{1}{4}(2\nu+1)\pi = 573^\circ-88^\circ\,17' = 484^\circ\,43'$, $\sqrt{(\frac{1}{2}\pi r)} = 4{\cdot}19$,

$$\zeta_{\frac{1}{3}}(z) \doteqdot 1, \quad \xi_{\frac{1}{3}}(z) = \frac{4\nu^2-1}{8re^{i\theta}} = \frac{-5e^{-i\theta}}{72r} = \frac{-5r}{72r^2}(\cos\theta - i\sin\theta)$$

$$= \frac{-5}{72\times 125}(10-5i) = -(5{\cdot}56-2{\cdot}78i)10^{-3}.$$

Thus

$$\zeta_{\frac{1}{3}}(z)+i\xi_{\frac{1}{3}}(z) = 1-(2{\cdot}78+5{\cdot}56i)10^{-3} \doteqdot e^{-2{\cdot}78\times 10^{-3}}e^{-5{\cdot}56\times 10^{-3}i}.$$

Now $5{\cdot}56^c\times 10^{-3} = 19'$, so

$$H^{(1)}_{\frac{1}{3}}(z) = \frac{e^{-r\sin\theta-2{\cdot}78\times 10^{-3}}}{4{\cdot}19}e^{i\{484^\circ 43'-19'\}} = \frac{e^{-4{\cdot}9972}}{4{\cdot}19}\,\underline{|484^\circ\,24'}$$

$$= 1{\cdot}609\times 10^{-3}(-\cos 55^\circ\,36'+i\sin 55^\circ\,36')$$

$$= 1{\cdot}609\times 10^{-3}(-0{\cdot}565+0{\cdot}8251i)$$

$$= -9{\cdot}09\times 10^{-4}+1{\cdot}33\times 10^{-3}i. \tag{68}$$

$$\zeta_{\frac{1}{3}}(z)-i\xi_{\frac{1}{3}}(z) = 1+(2{\cdot}78+5{\cdot}56i)10^{-3} \doteqdot e^{2{\cdot}78\times 10^{-3}}e^{5{\cdot}56\times 10^{-3}i},$$

so
$$H^{(2)}_{\frac{1}{3}}(z) = \frac{e^{5+2{\cdot}78\times 10^{-3}}}{4{\cdot}19}e^{-i\{573^\circ+13^\circ 17'-75^\circ-19'\}}$$

$$= 35{\cdot}36\,\overline{|510^\circ\,58'} = 35{\cdot}36\,\underline{|209^\circ\,2'}$$

$$= -35{\cdot}36\{0{\cdot}8743+0{\cdot}4853i\}$$

$$= -30{\cdot}9-17{\cdot}16i. \tag{69}$$

From (56) and (57)

$$J_{\frac{1}{3}}(z) = \tfrac{1}{2}\{H^{(1)}_{\frac{1}{3}}(z)+H^{(2)}_{\frac{1}{3}}(z)\} \doteqdot \tfrac{1}{2}H^{(2)}_{\frac{1}{3}}(z) = -15{\cdot}45-8{\cdot}58i. \tag{70}$$

$$Y_{\frac{1}{3}}(z) \doteqdot \tfrac{1}{2}iH^{(2)}_{\frac{1}{3}}(z) = 8{\cdot}58-15{\cdot}45i, \tag{71}$$

since $H^{(1)}_{\frac{1}{3}}(z)$ is negligible. From (18 a)

$$J_{\frac{1}{3}}(-z) = e^{-\frac{1}{3}\pi i}J_{\frac{1}{3}}(z) \doteqdot \tfrac{1}{2}e^{-\frac{1}{3}\pi i}H^{(2)}_{\frac{1}{3}}(z)$$

$$= 17{\cdot}68e^{i(209^\circ 2'-60^\circ)} = 17{\cdot}68\,\underline{|149^\circ\,2'}$$

$$= 17{\cdot}68\{-\cos 30^\circ\,58'+i\sin 30^\circ\,58'\}$$

$$= 17{\cdot}68\{-0{\cdot}8575+0{\cdot}5145i\}$$

$$= -15{\cdot}15+9{\cdot}1i. \tag{72}$$

From (19a)

$$
\begin{aligned}
Y_{\frac{1}{3}}(-z) &= e^{\frac{1}{3}\pi i}Y_{\frac{1}{3}}(z)-iJ_{\frac{1}{3}}(z) \\
&\doteqdot \tfrac{1}{2}e^{\frac{1}{3}\pi i}e^{\frac{1}{3}\pi i}H^{(2)}_{\frac{1}{3}}(z)-iJ_{\frac{1}{3}}(z) \quad \text{from (57), since } H^{(1)}_{\frac{1}{3}}(z) \text{ is negligible,} \\
&= \tfrac{1}{2}e^{\frac{2}{3}\pi i}H^{(2)}_{\frac{1}{3}}(z)-iJ_{\frac{1}{3}}(z) \\
&= 17\cdot 68e^{i(209^\circ 2'+150^\circ)}-iJ_{\frac{1}{3}}(z) = 17\cdot 68\,\lfloor\overline{58'}-iJ_{\frac{1}{3}}(z) \\
&= 17\cdot 68\{\cos 58'-i\sin 58'\}-iJ_{\frac{1}{3}}(z) \\
&= 17\cdot 68\{0\cdot 9999-0\cdot 0169i\}-iJ_{\frac{1}{3}}(z) \\
&= 17\cdot 68-0\cdot 299i-8\cdot 58+15\cdot 45i \\
&= 9\cdot 1+15\cdot 15i. \qquad (73)
\end{aligned}
$$

The results are, therefore,

$$
\left.
\begin{aligned}
J_{\frac{1}{3}}(10+5i) &= -15\cdot 45-8\cdot 58i \\
Y_{\frac{1}{3}}(10+5i) &= 8\cdot 58-15\cdot 45i \\
J_{\frac{1}{3}}(-10-5i) &= -15\cdot 15+9\cdot 1i \\
Y_{\frac{1}{3}}(-10-5i) &= 9\cdot 1+15\cdot 15i
\end{aligned}
\right\} \qquad (74)
$$

This example has been worked out in considerable detail in order to show the method of computation. The results were obtained by aid of a 10-inch slide rule, and are sufficiently accurate for the purpose of illustration.

13. Bessel loud-speaker horns [2]

The general differential equation for the propagation of sound waves in a very long loud-speaker horn with a linear axis is, for the steady state

$$\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}\frac{d\log A}{dx}+k^2\phi = 0, \qquad (75)$$

provided the sound pressure varies sinusoidally everywhere, which necessitates an extremely small amplitude [83]. ϕ is the velocity potential, this being a measure of the sound pressure at a definite frequency since $p = \rho_0\dfrac{\partial\phi}{\partial t} = i\rho_0\omega\phi$, ρ_0 being the normal air density: A is the cross-sectional area of the horn at any abscissa $x+x_0$, x_0 being the distance from the throat to the vertex (fictitious). The diaphragm is situated at the throat and vibrates sinusoidally with constant amplitude. $k = \omega/c = 2\pi/\lambda$ the phase constant, c being the velocity of sound and λ its wave-length at a frequency $\omega/2\pi$ (see p. xi).

Referring to Fig. 14, let the expansion curve of the horn be

$A = B_0(x+x_0)^\mu$, where μ the flaring index is a real positive quantity and B_0 is a constant. Then

$$\log A = \log B_0 + \mu \log(x+x_0),$$

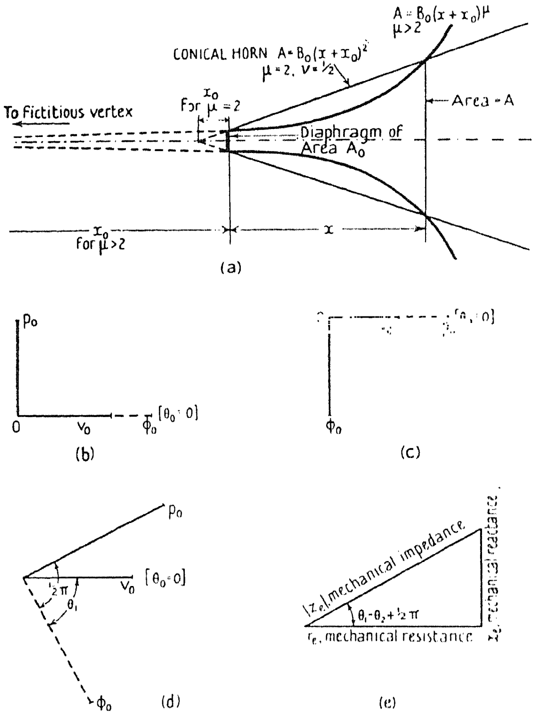


FIG. 14. (*a*) Diagrammatic representation of loud-speaker horn. (*b*) Phase relationship between velocity potential ϕ_0, particle velocity v_0, and sound pressure p_0 at the diaphragm of a loud-speaker horn when $kx_0 \ll 1$. (*c*) As at (*b*) when $kx_0 \gg 1$. (*d*) As at (*b*) and (*c*), but for intermediate values of kx_0. (*e*) Vector diagram of mechanical impedance due to air set in vibration at diaphragm. $\cos(\theta_1 - \theta_2 + \frac{1}{2}\pi) = r_e/z_e$ is the acoustical power factor. $(\theta_1 - \theta_2 + \frac{1}{2}\pi)$ is the angle between p_0 and v_0 in (*d*), since $\theta_0 = \theta_2 = 0$ and θ_1 is negative.

so
$$\frac{d \log A}{dx} = \mu/(x+x_0) = \mu/y, \tag{76}$$
where $y = x+x_0$.

Substituting from (76) in (75) we obtain

$$\frac{d^2\phi}{dy^2} + \frac{\mu}{y}\frac{d\phi}{dy} + k^2\phi = 0, \tag{77}$$

which is seen to be a form of Bessel equation. To solve (77) assume

$\phi = zy^{-\nu}$. Then differentiating with respect to y we get

$$\phi' = y^{-\nu}z' - \nu y^{-\nu-1}z,$$

so

$$\frac{\mu\phi'}{y} = \frac{\mu}{y^\nu}\left\{\frac{z'}{y} - \frac{\nu z}{y^2}\right\}. \tag{78}$$

Also

$$\phi'' = y^{-\nu}z'' - 2\nu y^{-\nu-1}z' + \nu(\nu+1)y^{-\nu-2}z$$
$$= \frac{1}{y^\nu}\left\{z'' - \frac{2\nu z'}{y} + \frac{\nu(\nu+1)z}{y^2}\right\}. \tag{79}$$

Adding $k^2\phi = \dfrac{k^2 z}{y^\nu}$ to the sum of (78) and (79) to reproduce (77), and multiplying throughout by y^ν, we obtain

$$z'' + (\mu - 2\nu)\frac{z'}{y} + \left[k^2 + \frac{\nu^2+\nu-\nu\mu}{y^2}\right]z = 0. \tag{80}$$

Putting $\mu - 2\nu = 1$, we get $\mu = 2\nu+1$ and $\nu^2+\nu-\nu\mu = -\nu^2$. Thus (80) becomes

$$\frac{d^2z}{dy^2} + \frac{1}{y}\frac{dz}{dy} + \left(k^2 - \frac{\nu^2}{y^2}\right)z = 0. \tag{81}$$

From § 2, Chap. II, the solution of (81) is seen to be one of the forms specified in (13) to (17) with argument ky. We shall take it to be

$$z = A_1 H_\nu^{(1)}(ky) + B_1 H_\nu^{(2)}(ky). \tag{82}$$

Since $\phi = zy^{-\nu}$ we obtain

$$\phi = \frac{1}{y^\nu}\{A_1 H_\nu^{(1)}(ky) + B_1 H_\nu^{(2)}(ky)\}, \tag{83}$$

where $y = (x+x_0)$, and $\nu = \frac{1}{2}(\mu-1)$.†

Determination of the constants A_1 and B_1 in (83).

Consider the value of ϕ at a distance from the fictitious vertex where $k(x+x_0) \gg 1$. Substituting the asymptotic values of the Bessel functions from (58) and (60) in (83) the solution can be written

$$\phi = \frac{1}{y^\nu}\sqrt{\left(\frac{2}{\pi ky}\right)}\{A_2 e^{ikx} + B_2 e^{-ikx}\}, \tag{84}$$

where $A_2 = A_1 e^{[kx_0-(2\nu+1)\pi/4]i}$ and $B_2 = B_1 e^{-[kx_0-(2\nu+1)\pi/4]i}$. Now as x increases, $e^{-ikx} = \cos kx - i\sin kx$ the distance phase factor, represents a vector rotating in a negative direction (clockwise), so that if $x_1 > x_2$ the phase at x_1 lags on that at x_2. Hence the second term in (84) represents a diverging wave travelling down the horn in the positive direction of x. In like manner the first term in (84) represents

† It is of interest to remark that when $\nu = -\frac{1}{2}$, the horn becomes a cylinder, so x_0, and therefore y, is very great and ϕ is independent of x, i.e. it is constant.

a converging wave travelling up the horn in the negative direction of x, which can be regarded as a reflected wave. We shall assume the conditions are such that the latter wave can be left out of account. Hence from (83) the solution of (77) for our requirements is

$$\phi = \frac{B_1}{y^\nu} H_\nu^{(2)}(ky). \tag{85}$$

When there is a reflected wave, both terms in (83) must be used. It is sometimes convenient, however, to use the form

$$\phi = \frac{1}{y^\nu}\{A_1 J_\nu(ky) + B_1 Y_\nu(ky)\} \tag{86}$$

as shown in example 74 at the end of this chapter.

To determine B_1 in (85) we have from (56) Chap. II, the air particle velocity

$$-\frac{\partial\phi}{\partial y} = -B_1\{ky^{-\nu}H_\nu^{(2)\prime}(ky) - \nu y^{-\nu-1}H_\nu^{(2)}(ky)\}. \tag{87}$$

Using the recurrence formula (74), p. 162, we obtain

$$v = -\frac{\partial\phi}{\partial y} = \frac{B_1 k H_{\nu+1}^{(2)}(ky)}{y^\nu}. \tag{88}$$

At the diaphragm $y = x_0$, $v = v_0$, so

$$B_1 = v_0 x_0^\nu / k H_{\nu+1}^{(2)}(kx_0)$$
$$= K_0 x_0^\nu e^{-i\theta_0}, \tag{89}$$

where $K_0 = v_0/k|H_{\nu+1}^{(2)}(kx_0)|$, $\theta_0 = \tan^{-1}\left\{-\frac{Y_{\nu+1}(kx_0)}{J_{\nu+1}(kx_0)}\right\}$, and

$$|H_{\nu+1}^{(2)}(kx_0)| = \sqrt{\{J_{\nu+1}^2(kx_0) + Y_{\nu+1}^2(kx_0)\}}.$$

Thus from (85) and (89), the velocity potential at the diaphragm where $y = x_0$ is

$$\phi_0 = K_0 H_\nu^{(2)}(kx_0)e^{-i\theta_0} = K_0|H_\nu^{(2)}(kx_0)|e^{i(\theta_1-\theta_0)}, \tag{90}$$

and $\theta_1 = \tan^{-1}\{-Y_\nu(kx_0)/J_\nu(kx_0)\}$. From (88) and (89) the velocity of the diaphragm

$$v_0 = K_0 k|H_{\nu+1}^{(2)}(kx_0)|e^{i(\theta_2-\theta_0)} \tag{91}$$

and $(y = x_0)$ $\theta_2 = \theta_0 = \tan^{-1}\{-Y_{\nu+1}(kx_0)/J_{\nu+1}(kx_0)\}$. Now the sound pressure is $p = i\rho_0\omega\phi$ (Chap. II, § 7), so we obtain from (90)

$$p_0 = \rho_0\omega K_0|H_\nu^{(2)}(kx_0)|e^{i(\theta_1-\theta_0+\frac{1}{2}\pi)}. \tag{92}$$

When $kx_0 \ll 1$, $Y_\nu(kx_0)$ and $Y_{\nu+1}(kx_0) \to -\infty$, so θ_0, θ_1, and $\theta_2 \to \frac{1}{2}\pi$. In practice this entails a very low frequency and a horn where x_0

is small, i.e. the diaphragm is near the fictitious vertex. If for convenience the phase of v_0 is taken as zero, that of ϕ_0 is also zero whilst that of p_0 is $\frac{1}{2}\pi$ as shown in Fig. 14B. Since p_0 and v_0 are in quadrature, no power is delivered to the horn, and the pressure on the diaphragm is due solely to the inertia of the air [83].

When $kx_0 \gg 1$ and $\nu = \frac{1}{2}(\mu-1)$ is such that $\zeta_\nu(kx_0) \doteqdot 1$ and $\xi_\nu(kx_0) \doteqdot 0$, $\theta_0 = \theta_2 = -kx_0+\frac{1}{4}\pi+\frac{1}{2}(\nu+1)\pi$, $\theta_1 = -kx_0+\frac{1}{4}\pi+\frac{1}{2}\nu\pi$, so $\theta_1-\theta_0 = -\frac{1}{2}\pi$; $\theta_2-\theta_0 = 0$, $\theta_1-\theta_0+\frac{1}{2}\pi = 0$. Thus p_0 and v_0 are in phase, whilst ϕ_0 lags by $\frac{1}{2}\pi$ as shown in Fig. 14C. The power P delivered to the horn is a maximum since the power factor or cosine of the angle between p_0 and v_0 is unity.

At intermediate values of kx_0, $\theta_0 = \theta_2$ and the phase relationships are illustrated in Fig. 14D. Since p_0 and v_0 are not in phase, the power delivered to the horn is now reduced to $P\cos(\theta_1-\theta_2+\frac{1}{2}\pi)$, where $\cos(\theta_1-\theta_2+\frac{1}{2}\pi) < 1$. As either the distance of the diaphragm from the fictitious vertex of the horn, or the frequency, or both, increase, ν being constant meanwhile, $\theta_1-\theta_2$ approaches $-\frac{1}{2}\pi$ and Fig. 14C is obtained as before.

It will be seen from the asymptotic expansions for $H_\nu^{(2)}(kx_0)$ and $H_{\nu+1}^{(2)}(kx_0)$ that the greater ν, the greater must be kx_0 to make

$$\zeta_{\nu+1}(kx_0) \doteqdot 1 \quad \text{and} \quad \xi_{\nu+1}(kx_0) \doteqdot 0.$$

Thus if the flaring index of the horn $\mu = 2\nu+1$ is increased, the value of x_0 must also be correspondingly increased if it is desired to keep p and v in phase. Since the diameter of the diaphragm is constant, this increase in x_0 occurs automatically.

14. Relative performance of Bessel horns of different orders ν

The mechanical impedance z_e due to the air set in vibration by the diaphragm consists of two components in quadrature as shown in Fig. 14E. Thus

$$z_e = r_e+ix_e, \tag{93}$$

where r_e is the resistive or load component due to sound radiation, and x_e is the inertia or wattless component due to fluid inertia [83]. The higher the resistance relative to $|z_e|$, the greater the efficiency and the better the performance of the horn. In an alternating current circuit, impedance = e.m.f./current, so in the mechanical case $z_e = p_0 A_0/v_0$, where p_0 is the sound pressure per unit area on the diaphragm, whose area is A_0 and axial velocity v_0.

From (91), (92)

$$z_e = p_0 A_0/v_0 = r_e + ix_e = \rho_0 c A_0 \frac{|H_\nu^{(2)}(kx_0)|}{|H_{\nu+1}^{(2)}(kx_0)|} e^{i(\theta_1-\theta_2+\frac{1}{2}\pi)}, \tag{94}$$

where $c = \omega/k$, so

$$r_e = \rho_0 c A_0 \frac{|H_\nu^{(2)}(kx_0)|}{|H_{\nu+1}^{(2)}(kx_0)|} \cos(\theta_1-\theta_2+\tfrac{1}{2}\pi) \tag{94a}$$

and

$$x_e = \rho_0 c A_0 \frac{|H_\nu^{(2)}(kx_0)|}{|H_{\nu+1}^{(2)}(kx_0)|} \sin(\theta_1-\theta_2+\tfrac{1}{2}\pi). \tag{94b}$$

To compare the performances of horns with different rates of expansion, we fix the initial and final areas A_0, A_l, also l, the distance between them, which must be such that the radius at the mouth is not less than one-quarter the length of the longest wave to be reproduced. This latter condition is imposed in order to reduce reflection at the mouth to a negligible amount.

We have $A_0 = B_0 x_0^\mu$; $A_l = B_0(x_0+l)^\mu$, so $\dfrac{x_0+l}{x_0} = \left(\dfrac{A_l}{A_0}\right)^{1/\mu} = b^{1/\mu}$, where $b = A_l/A_0$. Thus

$$x_0 = l/(b^{1/\mu}-1). \tag{95}$$

From (95) it is seen that x_0 increases with increase in μ, so the distance from the fictitious vertex to the diaphragm is augmented as shown in Fig. 14A. By inserting the value of x_0 found from (95) in (94) and taking a series of values of frequency, i.e. of $k = \omega/c$, it is found that at low frequencies r_e rises relative to $|z_e|$ with increase in ν and therefore in $\mu = 2\nu+1$. Thus the performance of the horn in the lower musical register improves with increase in the flaring index μ. Curves illustrating this feature and calculated from (94a) are reproduced in Fig. 15.

From an analytical point of view it is important to realize that the performance of a horn of great length (from A_0 to A_l), using a diaphragm of fixed radius, depends upon x_0. From above we see that the greater x_0 the higher the relative value of r_e, the mechanical resistance at the throat. If in Fig. 14A we imagine a conical horn of very small apical angle, to be contained wholly within the horn of large flaring index, the former will have a slight advantage over the latter [83], provided the diaphragm is the same size in both cases. But in practice, where the length of the horn must be restricted on account of economy of space and cost, a flared horn is vastly superior to a conical horn at the lower frequencies. In making this comparison,

the respective lengths, throat and mouth areas are equal in both cases.

The effect of flaring is to introduce a very gradual change in the impedance offered to the sound waves, particularly in the neighbourhood of the diaphragm.† In this way x_e the wattless or inertia component of the mechanical impedance is reduced, and r_e the load or resistive component is increased.

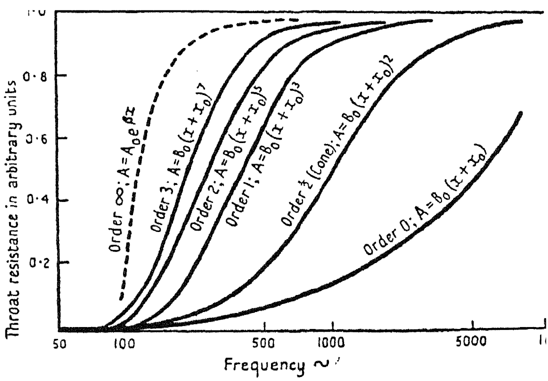


FIG. 15. Mechanical resistance curves for Bessel horns of various orders. At a given frequency r_e increases with the flaring index $\mu = 2\nu+1$.

$r_e = \rho_0 cA_0[J_{\nu+1}(kx_0)Y_\nu(kx_0)-J_\nu(kx_0)Y_{\nu+1}(kx_0)]/|H^{(2)}_{\nu+1}(kx_0)|^2;$

$x_e = \rho_0 cA_0[J_\nu(kx_0)J_{\nu+1}(kx_0)+Y_\nu(kx_0)Y_{\nu+1}(kx_0)]/|H^{(2)}_{\nu+1}(kx_0)|^2.$

Formula (94) can be written

$$z_e = \rho_0 cA_0 \frac{H^{(2)}_\nu(kx_0)}{H^{(2)}_{\nu+1}(kx_0)} e^{\frac{1}{2}\pi i}. \tag{96}$$

When kx_0 is large we have from (60)

$$H^{(2)}_\nu(kx_0)$$
$$= \sqrt{\left(\frac{2}{\pi kx_0}\right)} e^{-i(kx_0-\frac{1}{4}\pi-\frac{1}{2}\nu\pi)}\left\{1+\frac{4\nu^2-1^2}{1!(8kx_0 i)}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8kx_0 i)^2}+\ldots\right\}. \tag{97}$$

If ν is very large compared with unity the bracketed series is substantially equal for ν and $\nu+1$. Consequently (96) degenerates to

$$z_e = r_e = \rho_0 cA_0 \tag{98}$$

so the reactive component vanishes, the mechanical impedance is wholly resistive and the power factor is unity.

In practice the expansion curve of a horn is usually of the form

† The impedance of a uniform tube of great length is entirely resistive.

$A = A_0 e^{\beta x}$ and it is, therefore, of considerable interest to show that when the flaring index of a Bessel horn $\mu \to \infty$, the expansion curve becomes exponential, provided a certain condition is satisfied. Taking formula (95), we have

$$b^{1/\mu} = 1+\frac{\log b}{\mu}+\frac{1}{2!}\left(\frac{\log b}{\mu}\right)^2+\ldots, \quad \text{so} \quad \lim_{\mu\to\infty}\mu(b^{1/\mu}-1) = \log b$$

and
$$x_0 = \mu l/\log b = \mu/\beta, \tag{99}$$
where $\beta = \dfrac{\log b}{l}$ = a constant, since in the above comparison with varying μ, l is fixed. From preceding work $B_0 = A_0/x_0^\mu$, so $A = A_0(1+x/x_0)^\mu$. Substituting for x_0 from (99) we obtain

$$A = \lim_{\mu\to\infty} A_0\left(1+\frac{\beta x}{\mu}\right)^\mu = A_0 e^{\beta x}, \tag{100}$$

where β is the flaring index of the exponential horn. It will be seen that x_0, the distance of the diaphragm from the fictitious vertex, is now infinite.

EXAMPLES

1. Evaluate $F(-\frac{1}{2}, \frac{3}{2}, 3, 1)$ and check by means of the series. [$32/15\pi$.]

2. Show that

(a) $F(1,1,2,-z) = (1/z)\log(1+z)$; (b) $F(-n,1,1,-z) = (1+z)^n$;

(c) $F(\alpha,\beta,\beta,z) = (1-z)^{-\alpha}$; (d) $\lim_{\beta\to\infty} F(1,\beta,1,z/\beta) = e^z$;

(e) $zF(\frac{1}{2},\frac{1}{2},\frac{3}{2},z^2) = \sin^{-1}z$; (f) $zF(\frac{1}{2},1,\frac{3}{2},-z^2) = \tan^{-1}z$;

(g) If $F(\gamma;z) = 1+\dfrac{z}{\gamma}+\dfrac{z^2}{2!\gamma(\gamma+1)}+\dfrac{z^3}{3!\gamma(\gamma+1)(\gamma+2)}\ldots$,

show that $J_\nu(z) = \dfrac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}F(\nu+1; -\frac{1}{4}z^2)$.

3. Show that $$F(\alpha,\beta,\gamma,1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}F(\alpha,\beta,\gamma+1,1).$$

4. Evaluate $$\int_0^1 F(-\tfrac{1}{2},\tfrac{1}{2},1,b^2)b\,db.$$ [$4/3\pi$.]

5. Evaluate the following integral which occurs in an acoustical problem [5]:
$$\int_0^1 F(\alpha,\beta,2,b^2)b^3\,db \quad \text{when } \alpha = -\tfrac{3}{2}, \beta = \tfrac{1}{2}.$$ [$2^6/105\pi$.]

6. Show [8] that $$\int_0^\pi (1-b^2\sin^2\theta)^{\frac{1}{2}}\,d\theta = \pi F(-\tfrac{1}{2},\tfrac{1}{2},1,b^2).$$ [Expand and integrate.]

7. Show [8] that
$$\int_0^\pi \{b\cos\theta+(1-b^2\sin^2\theta)^{\frac{1}{2}}\}^3\,d\theta = \pi\{F(-\tfrac{3}{2},\tfrac{1}{2},1,b^2)+\tfrac{3}{2}b^2F(-\tfrac{1}{2},\tfrac{1}{2},2,b^2)\}.$$
[Expand and integrate.]

8. Show [8] that $\int\limits_0^\pi \{b\cos\theta+(1-b^2\sin^2\theta)^{\frac{1}{2}}\}^n\,d\theta$

$$=\pi\sum_{p=1,3,\ldots n}\frac{n(n-1)\ldots(n-p-1)}{p!}\,\frac{(n-p-1)\ldots 1}{(n-p)\ldots 2}\,b^{n-p}F\Big(-p/2,\tfrac{1}{2},\frac{n-p+2}{2},b^2\Big)$$

when n is an odd positive integer. The integrals in 6, 7, and 8 occur in finding the pressure at any point on a rigid disk vibrating in an infinite rigid plane.

9. Evaluate $\int\limits_0^a (1-z^2/a^2)zF(-\tfrac{3}{2},\tfrac{1}{2},1,z^2/a^2)\,dz.$

[$\tfrac{1}{4}a^2F(-\tfrac{3}{2},\tfrac{1}{2},3,1) = 2^6a^2/105\pi$.]

10. Verify that

(a) $\Gamma(\tfrac{11}{2}) = \dfrac{9!\sqrt{\pi}}{4!\,2^9}$; (b) $\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$; (c) $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin\pi z$; (d) $\Gamma(\tfrac{2}{3})\Gamma(\tfrac{1}{3}) = 2\pi/\sqrt{3}$; (e) $\Gamma(n+\tfrac{1}{2}) = \dfrac{(2n-1)!\sqrt{\pi}}{(n-1)!\,2^{2n-1}}$, when n is a positive integer; (f) $\dfrac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} = 1.3.5\ldots(2n-1)$; (g) $\Gamma(z)\Gamma(-z) = -\pi/z\sin\pi z$;

(h) $\Gamma(-\tfrac{5}{2}) = -\dfrac{8}{15}\sqrt{\pi}$.

11. Verify that (a) $\Gamma(2z) = \dfrac{(2z-1)!\,\Gamma(z)}{(z-1)!}$, z a positive integer;

(b) $\Gamma(2z) = \dfrac{2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})}{\sqrt{\pi}}$;

(c) $\dfrac{2\sin\nu\pi}{\pi} = \dfrac{1}{\Gamma(\nu)\Gamma(1-\nu)}-\dfrac{1}{\Gamma(1+\nu)\Gamma(-\nu)}$.

12. Evaluate $\Gamma(\tfrac{1}{3})$ and $\Gamma(\tfrac{2}{3})$. [2·679; 1·354. Use formula (15), put $\alpha = -\tfrac{2}{3}$, $\beta = -\tfrac{1}{3}$, $\gamma = \tfrac{10}{3}$, and calculate the value of the hypergeometric series. Then use the result in example 10 (d). The value of γ can be $\tfrac{1}{3}(n+1)$, where n is a positive integer and $n+1$ is not divisible by 3. The value chosen gives fairly rapid convergence of the hypergeometric series.]

13. (a) Given that $\sin z = z\prod\limits_{m=1}^{\infty}\Big(1-\dfrac{z^2}{m^2\pi^2}\Big)$, prove formula (12) in the text.

[Write πz for z and use formula (7) and example 10 (g).]

(b) Prove that $e^{1-\gamma}\prod\limits_{m=2}^{\infty}\Big\{\Big(1-\dfrac{1}{m}\Big)e^{1/m}\Big\} = 1$, where γ is Euler's constant.

(c) Show that $\int\limits_0^\infty e^{-t}t^{z-1}\,dt = \dfrac{1}{z}\int\limits_0^\infty e^{-t}t^z\,dt.$ [Use (10 a) in § 2.]

14. Evaluate (a) $\int\limits_0^\infty e^{-x^2}\,dx$; (b) $\int\limits_0^\infty e^{-x^n}\,dx$.

[(a) $\tfrac{1}{2}\sqrt{\pi}$, use (10 a) in § 2, put $t = x^2$ and $z = \tfrac{1}{2}$; (b) $\Gamma\Big(1+\dfrac{1}{n}\Big)$.]

15. Show that (a) $\dfrac{\Gamma(\nu+r+\frac{1}{2})}{r!\,\Gamma(\nu-r+\frac{1}{2})} = \dfrac{(4\nu^2-1^2)(4\nu^2-3^2)\ldots\{4\nu^2-(2r-1)^2\}}{2^{2r}r!}$;

$$(b)\ \int_0^\infty e^{-tr}t^z\,dt = \frac{\Gamma(1+z)}{r^{1+z}}.\quad R(z) > -1.$$

(c) If p and q are positive integers, prove that [77]

$$2^{p+q-1}\Gamma\frac{(p+q)}{2}\Gamma\Big(\frac{p+q+1}{2}\Big) = \Gamma(\tfrac{1}{2})\Gamma(p+q).$$

16. Evaluate (a) $\displaystyle\int_0^\pi e^{iz\cos\theta}(\cos 2\theta-1)\,d\theta.$ $\Big[-2\pi\dfrac{J_1(z)}{z}.\Big]$

(b) $\displaystyle\int_0^{\frac{1}{2}\pi} \cos(z\cos 2\theta)\,d\theta.$ $[\frac{1}{2}\pi J_0(z).]$

17. When z is complex show that $J_\nu(z)$ can be put in the form

$$J_\nu(z) = \frac{(\frac{1}{2}r)^\nu}{\Gamma(\nu+1)}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2}r)^{2m}}{m!(\nu+1)\ldots(\nu+m)}[\cos(\nu+2m)\theta+i\sin(\nu+2m)\theta]\right\}.$$

This is suitable for computation when z is not large enough for the asymptotic expansions to be used. Find the value of $J_{\frac{2}{3}}(1+i)$.

[See example 12 for the gamma function. A problem of this type occurs in tapered loaded cables, § 5, Chap. VII.]

18. Show that

(a) $J_{-\nu}(-z) = e^{-i\nu\pi}J_{-\nu}(z)$; (b) $Y_\nu(-z) = e^{i\nu\pi}Y_\nu(z)+2iJ_{-\nu}(z)$;

(c) $Y_{-\nu}(-z) = e^{-i\nu\pi}Y_{-\nu}(z)+2i\,J_\nu(z)$ $(-\pi<\theta\leqslant 0)$;

(d) $H_\nu^{(1)}(z) = -e^{-i\nu\pi}H_\nu^{(2)}(ze^{-i\pi})$ $(0\leqslant\theta\leqslant\pi)$.

19. Show that (a) $H_\nu^{(1)}(-z) = \dfrac{e^{-i\nu\pi}J_{-\nu}(z)-J_\nu(z)}{i\sin\nu\pi}$;

$$(b)\ H_\nu^{(2)}(-z) = \frac{i[e^{-i\nu\pi}J_{-\nu}(z)-e^{2i\nu\pi}J_\nu(z)]}{\sin\nu\pi}\quad(-\pi<\theta\leqslant 0).$$

20. Verify that (a) $H_\nu^{(2)}(re^{i\theta}) = e^{i\nu\pi}H_\nu^{(1)}\{re^{i(\theta-\pi)}\}+2\cos\nu\pi\,H_\nu^{(2)}\{re^{i(\theta-\pi)}\}$;

(b) $H_\nu^{(1)}(re^{i\theta}) = e^{-i\nu\pi}H_\nu^{(2)}\{re^{i(\theta+\pi)}\}+2\cos\nu\pi\,H_\nu^{(1)}\{re^{i(\theta+\pi)}\}$.

21. Given that $J_0(z) = \dfrac{1}{\pi}\displaystyle\int_0^\pi e^{iz\cos\theta}\,d\theta$, show by differentiating $J_0(z)$ and the integrand with respect to z, that $J_1(z) = -\dfrac{i}{\pi}\displaystyle\int_0^\pi e^{iz\cos\theta}\cos\theta\,d\theta$.

22. Verify that

$$(a)\ \int_0^a \frac{x/a}{\sqrt{(a^2-x^2)}}J_0(kx\sin\phi)\,dx = \sqrt{\Big(\frac{\pi}{2z}\Big)}J_{\frac{1}{2}}(z) = \sqrt{\Big(\frac{\pi}{2z}\Big)}\mathbf{H}_{-\frac{1}{2}}(z)\quad(z=ka\sin\phi);$$

$$(b)\ \int_0^1 (1-t^2)\cos zt\,dt = \sqrt{(2\pi)}z^{-\frac{3}{2}}J_{\frac{3}{2}}(z).$$

23. Prove that $$\mathbf{H}_0(z) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{J_{2n+1}(z)}{2n+1}.$$ [Use (7) or (15) Chap. III.]

24. Show that $$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left[\frac{1-\cos(z\sin\theta)}{\sin\theta}\right]d\theta = \int_0^z \mathbf{H}_0(z)\,dz.$$

25. Verify that $$\int^z \frac{\mathbf{H}_2(z)}{z}\,dz = \frac{2z}{3\pi}-\mathbf{H}_1(z)/z.$$

26. Show that $$\int^z z^3\mathbf{H}_0(z)\,dz = z^3\mathbf{H}_1(z)-2z^2\mathbf{H}_2(z).$$

27. Evaluate $$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-iz\cos\theta}\cos^2\theta\,d\theta.$$
$$\left[\pi\left\{\left[J_0(z)-\frac{J_1(z)}{z}\right]-i\left[\mathbf{H}_0(z)-\frac{\mathbf{H}_1(z)}{z}\right]\right\}.\right]$$

28. Evaluate $$\int_0^{\frac{1}{2}\pi} e^{-iz\cos\theta}\sin^2\theta\,d\theta.$$ $$\left[\frac{\pi}{2z}[J_1(z)-i\mathbf{H}_1(z)].\right]$$

29. Evaluate $$\int_0^{\frac{1}{2}\pi} e^{-3iz\cos\theta}\cos 2\theta\,d\theta.$$
$$\left[\tfrac{1}{2}\pi\left[J_0(3z)-\frac{2}{z}J_1(3z)\right]-\tfrac{1}{2}\pi i\left[\mathbf{H}_0(3z)-\frac{2}{z}\mathbf{H}_1(3z)\right].\right]$$

30. Evaluate $$\int_0^{\frac{1}{2}\pi} e^{iz\cos\theta}\sin\theta\,J_0(b\sin\theta)\,d\theta.$$
$$\left[\sqrt{\left(\frac{\pi}{2}\right)}\sum_{r=0}^{\infty}\frac{b^{2r}}{2^r r!\,z^{r+\frac{1}{2}}}[J_{r+\frac{1}{2}}(z)+i\mathbf{H}_{r+\frac{1}{2}}(z)].\right]$$

31. Plot $z\mathbf{H}_1(z)$ from $z = 0$ to 12, using the tabular values on p. 176.

32. Solve $\dfrac{d^2y}{dz^2}\pm a^2y = 0$ in terms of Bessel functions.

$[z^{\frac{1}{2}}[A_1 J_{\frac{1}{2}}(az)+B_1 J_{-\frac{1}{2}}(az)]$; $z^{\frac{1}{2}}[A_1 J_{\frac{1}{2}}(azi)+B_1 J_{-\frac{1}{2}}(azi)]$.

33. Plot $J_{\frac{1}{2}}(z)$ from $z = 0$ to 8 using sine tables.

34. Plot $J_{-\frac{1}{2}}(z)$ from $z = 0$ to 8 using cosine tables.

35. Express $J_{\frac{5}{2}}(z)$ in terms of circular functions. $\left[\left(\dfrac{3}{z^2}-1\right)\sin z-\dfrac{3}{z}\cos z.\right]$

36. Write down the series for $J_{\frac{1}{3}}(z)$. Calculate the values of $J_{\frac{1}{3}}(1)$ and $J_{\frac{1}{3}}(2)$.
$$\left[J_{\frac{1}{3}}(z) = \frac{(\frac{1}{2}z)^{\frac{1}{3}}}{\Gamma(\frac{4}{3})}\left\{1-\frac{3}{4}(\tfrac{1}{2}z)^2+\frac{3^2}{2!\,4.7}(\tfrac{1}{2}z)^4-\frac{3^3}{3!\,4.7.10}(\tfrac{1}{2}z)^6+\ldots\right\};\ 0{\cdot}7309;\ 0{\cdot}4429.\right.$$
See example 12 for $\Gamma(\frac{1}{3})$.]

37. Prove that (a) $$z^{\frac{3}{2}}J_{\frac{3}{2}}(z) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{d}{dz}(\sin z-z\cos z);$$

(b) $$\int_0^{\infty} e^{-z}J_0(\beta z)\,dz = \sqrt{\left[\left(\frac{\pi\beta}{2}\right)\int_0^{\infty} e^{-z}J_{-\frac{1}{2}}(\beta z)\,dz\right]}.$$

[See example 35 (b), Chap. I.]

38. Find the least positive zero of $J_{-\frac{1}{3}}(z)$ by plotting a curve. [1·87.]

39. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+\left(1-\dfrac{1}{z^2}\right)y = \dfrac{2}{\pi}$. [$y = A_1 J_1(z)+B_1 Y_1(z)+\mathbf{H}_1(z)$.]

40. Show that the maximum and minimum values of $z\mathbf{H}_1(z)$ correspond to the zeros of $\mathbf{H}_0(z)$.

41. The accession to inertia, this being the added mass due to a fluid, of a rigid disk vibrating axially in a very large flat wall is $m_i = 2\pi\rho_0 a^3\left\{\dfrac{\mathbf{H}_1(2z)}{z^2}\right\}$, where a = radius, ρ_0 = density of fluid, $z = ka$, $k = \omega/c$, c = velocity of sound, $\omega/2\pi$ = frequency. Show that, if a is constant, the maximum and minimum values of m_i are given by $3\mathbf{H}_1(2z) = z\mathbf{H}_0(2z)$.

42. Verify that the maximum and minimum values of $\mathbf{H}_0(z)$ correspond to the roots of $\mathbf{H}_1(z) = \dfrac{2}{\pi}$ (see Fig. 13), and that the zeros of $\mathbf{H}_0(z)$ occur when

$$z = \tfrac{1}{2}\pi \int\limits_0^z \mathbf{H}_1(z)\,dz.$$

43. If $\mathbf{H}_0(z) = 0{\cdot}00494$ and $\mathbf{H}_1(z) = 1{\cdot}0113$ when $z = 4{\cdot}32$, show that $\mathbf{H}_0(z)$ has a zero at $z = 4{\cdot}3332$.
[Use Taylor's theorem $f(z+h) \doteq f(z)+hf'(z)$ when $h \ll 1$.]

44. Show that
(*a*) the points of inflexion of $\mathbf{H}_0(z)$ correspond to the maximum and minimum values of $\mathbf{H}_1(z)$. See Fig. 13.
(*b*) $\mathbf{H}_1(z)$ represents $\dfrac{1}{z}$ times the area $\int\limits_0^z z\mathbf{H}_0(z)\,dz$, i.e. that $\mathbf{H}_1(z)$ is the mean value of $z\mathbf{H}_0(z)$ over the interval 0 to z.

45. Show that $\mathbf{H}_n(z)$ is a solution of [93]

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y-\frac{(\frac{1}{2}z)^{n-1}}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} = 0.$$

Show that $y = A_1 J_n(z)+B_1 Y_n(z)+\mathbf{H}_n(z)$ is also a solution, and write down the equation whose solution is $y = A_1 J_n(z)+B_1 Y_n(z)+C_1 \mathbf{H}_n(z)$.

46. Show that $J_{\frac{1}{2}}(z) = Y_{-\frac{1}{2}}(z) = \mathbf{H}_{-\frac{1}{2}}(z)$.

47. Show that the solution of $y''+\dfrac{1}{z}y'+(k^2-1/4z^2)y = 0$ can be written in the form $y = A_1 Y_{\frac{1}{2}}(kz)+B_1 \mathbf{H}_{-\frac{1}{2}}(kz)$.

48. Prove that $\mathbf{H}_{\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)}-J_{-\frac{1}{2}}(z)$

$$= \sqrt{\left(\frac{2z}{\pi}\right)} \int\limits_0^{\frac{1}{2}\pi} J_1(z\sin\theta)\,d\theta = \sqrt{\left(\frac{2}{\pi z}\right)}(1-\cos z).$$

49. Prove that

$$(a)\quad \int\limits_0^{\sqrt{(2z/\pi)}} \sin(\tfrac{1}{2}\pi\varpi^2)\,d\varpi = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^z \frac{\sin z}{z^{\frac{1}{2}}}\,dz$$

$$= \tfrac{1}{2}\int\limits_0^z J_{\frac{1}{2}}(z)\,dz = \sum_{n=0}^{\infty} J_{2n+\frac{3}{2}}(z).$$

$$(b) \int_0^{\sqrt{(2z/\pi)}} \cos(\tfrac{1}{2}\pi\varpi^2)\, d\varpi = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos z}{z^{\frac{1}{2}}}\, dz$$

$$= \tfrac{1}{2} \int_0^z J_{-\frac{1}{2}}(z)\, dz = \sum_{n=0}^{\infty} J_{2n+\frac{1}{2}}(z).$$

50. Show that, when z is real and $n > 0$,

$$(a) \lim_{z\to 0} \frac{J_{n+\frac{1}{2}}(z)}{z^{\frac{1}{2}}} = 0;$$

$$(b) \lim_{z\to \infty} \frac{J_{n+\frac{1}{2}}(z)}{z^{\frac{1}{2}}} = 0;$$

$$(c) \lim_{z\to 0} \frac{J_{-n-\frac{1}{2}}(z)}{z^{\frac{1}{2}}} = \infty;$$

$$(d) \lim_{z\to \infty} \frac{J_{-n-\frac{1}{2}}(z)}{z^{\frac{1}{2}}} = 0.$$

51. By expanding into series and integrating term by term, verify that

$$\int_0^{\infty} e^{-bx} x^n J_n(ax)\, dx = \frac{\Gamma(n+\frac{1}{2})(2a)^n}{\sqrt{\pi}(a^2+b^2)^{n+\frac{1}{2}}}.$$

Is any restriction required to make the integral convergent at the lower limit?

52. Show that [93]

$$(a) H^{(1)}_{n+\frac{1}{2}}(z) = \frac{2(\frac{1}{2}z)^{n+\frac{1}{2}}}{\sqrt{\pi}\, n!} \left\{\left(1+\frac{d^2}{dz^2}\right)^n \left(\frac{\sin z - i\cos z}{z}\right)\right\};$$

$$(b) H^{(2)}_{n+\frac{1}{2}}(z) = \frac{2(\frac{1}{2}z)^{n+\frac{1}{2}}}{\sqrt{\pi}\, n!} \left\{\left(1+\frac{d^2}{dz^2}\right)^n \left(\frac{\sin z + i\cos z}{z}\right)\right\}.$$

53. Use the asymptotic series to compute $J_3(12)$ and $Y_3(12)$. [0·1951; 0·129.]

54. Show that when $|z|$ is large enough and $-\frac{1}{2}\pi \leqslant \text{phase } z \leqslant \frac{1}{2}\pi$,

$$(a) J_0(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)} \cos(z - \tfrac{1}{4}\pi);$$

$$(b) Y_0(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)} \sin(z - \tfrac{1}{4}\pi); \quad \mathrm{Y}_0(z) \doteqdot \sqrt{\left(\frac{\pi}{2z}\right)} \sin(z - \tfrac{1}{4}\pi) +$$

$$+(\log 2 - \gamma) J_0(z) = \sqrt{\left(\frac{\pi}{2z}\right)} \sin(z - \tfrac{1}{4}\pi) + \sqrt{\left(\frac{2}{\pi z}\right)} (\log 2 - \gamma) \cos(z - \tfrac{1}{4}\pi).$$

55. When (z) is large enough and $-\frac{1}{2}\pi \leqslant \text{phase } z \leqslant \frac{1}{2}\pi$, show that

$$(a) H_0^{(1)}(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)} e^{i(z-\frac{1}{4}\pi)}; \quad (b) H_0^{(2)}(z) \doteqdot \sqrt{\left(\frac{2}{z\pi}\right)} e^{-i(z-\frac{1}{4}\pi)}.$$

56. When $|z|$ is large enough and $-\frac{1}{2}\pi \leqslant \text{phase } z \leqslant \frac{1}{2}\pi$, show that

$$(a) H_\nu^{(1)}(z) \doteqdot \frac{e^{-r\sin\theta}}{\sqrt{(\frac{1}{2}\pi r)}} e^{i\{r\cos\theta - \frac{1}{2}\theta - \frac{1}{4}(2\nu+1)\pi\}};$$

$$(b) H_\nu^{(2)}(z) \doteqdot \frac{e^{r\sin\theta}}{\sqrt{(\frac{1}{2}\pi r)}} e^{-i\{r\cos\theta + \frac{1}{2}\theta - \frac{1}{4}(2\nu+1)\pi\}}.$$

57. Using formulae (94 a) and (94 b) in the text, prove that when kx_0 is large enough $r_e \to \rho_0 c A_0$, whilst $x_e \to 0$. Is it necessary to restrict ν to obtain these results?

58. Show that, when $|z|$ is large enough,

$$(a)\ H_\nu^{(1)}(z)H_\nu^{(2)}(z) = \frac{2}{\pi z};\quad (b)\ \frac{H_\nu^{(1)}(z)}{H_{\nu+1}^{(1)}(z)} = i;\quad (c)\ \frac{H_\nu^{(2)}(z)}{H_{\nu+1}^{(2)}(z)} = -i.$$

If in formula (59) zi is written for z, show that the angular limits are $-\pi < \theta \leqslant 0$. [Write $(\frac{1}{2}\pi+\theta)$ for θ and rearrange.]

59. Using example 54, show that when z is real, the area included between the curve $J_0(z)$ and that portion of the x-axis between any two consecutive large roots θ_n, θ_{n+1} is approximately $\sqrt{\left(\frac{8}{\pi z_1}\right)}\cos(\theta_n+\frac{1}{4}\pi)$, $z_1 = \frac{1}{2}(\theta_n+\theta_{n+1})$.

60. When $|z|$ is large enough show that, approximately,

(a) $J_0(z) = (1/\sqrt{2})\{J_{\frac{1}{2}}(z)+J_{-\frac{1}{2}}(z)\}$; (b) $Y_0(z) = (1/\sqrt{2})\{J_{\frac{1}{2}}(z)-J_{-\frac{1}{2}}(z)\}$;

(c) $\dfrac{J_0(z)}{J_{-\frac{1}{2}}(z)} = \dfrac{1}{\sqrt{2}}(1+\tan z)$; (d) $\dfrac{J_0(z)}{J_{\frac{1}{2}}(z)} = \dfrac{1}{\sqrt{2}}(1+\cot z)$;

(e) $\dfrac{Y_0(z)}{Y_{-\frac{1}{2}}(z)} = \dfrac{1}{\sqrt{2}}(1-\cot z)$.

61. When both limits of integration are positive and large enough, show that

$$\int\limits_{z_2}^{z_1} z^{\frac{1}{2}}J_0(z)\,dz \doteqdot \left[\sqrt{\left(\frac{z}{2}\right)}\{J_{\frac{1}{2}}(z)-J_{-\frac{1}{2}}(z)\}\right]_{z_2}^{z_1}.$$

62. When $z > 10$ show that the roots of

$$J_\nu(z) = 0,\qquad Y_\nu(z) = 0,$$

$$J_{\nu+1}(z)Y_\nu(z)+J_\nu(z)Y_{\nu+1}(z) = 0,\quad\text{and}\quad J_0(\varphi z)Y_0(z)-J_0(z)Y_0(\varphi z) = 0,\ \varphi > 1,$$

are given approximately by the respective formulae:

$$z = \pi(m-\tfrac{1}{4}+\tfrac{1}{2}\nu);\ z = \pi(m+\tfrac{1}{4}+\tfrac{1}{2}\nu);\ z = \tfrac{1}{2}(m+\nu+1)\pi;\ z = m\pi/(\varphi-1),$$

where m is a positive integer.

63. Show that (a) $\mathbf{H}_0(z) \to 0$, (b) $\mathbf{H}_1(z) \to 2/\pi$ when $z \to \infty$. Also show that $\mathbf{H}_1(z) = 0$ has no positive root except zero. [Use asymptotic expansion.]

64. When z is large show that

$$H_\nu^{(1)}(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}e^{i(z-\frac{1}{4}\pi-\frac{1}{2}\nu\pi)}\left\{1-\frac{4\nu^2-1^2}{1!(8zi)}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8zi)^2}-\ldots\right\}$$

and

$$H_\nu^{(2)}(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}e^{-i(z-\frac{1}{4}\pi-\frac{1}{2}\nu\pi)}\left\{1+\frac{4\nu^2-1^2}{1!(8zi)}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8zi)^2}+\right.$$
$$\left.+\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{3!(8zi)^3}+\ldots\right\}.$$

65. Show that

(a) when z is in the second quadrant, the asymptotic expansion of $J_\nu(z)$ can be written $e^{(\nu+\frac{1}{2})\pi i}\sqrt{\left(\frac{2}{\pi z}\right)}\{\zeta_\nu(z)\cos(z+\frac{1}{4}\pi+\frac{1}{2}\nu\pi)-\xi_\nu(z)\sin(z+\frac{1}{4}\pi+\frac{1}{2}\nu\pi)\}$;

(b) when z is in the third quadrant this formula holds provided the sign of the exponential is changed.

[The asymptotic expansion (52) in § 11 holds for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, as stated. Diametrically opposite in the fourth quadrant we have $-z = ze^{-i\pi}$, where z is the value in the second quadrant. From (18 a) $J_\nu(ze^{-i\pi}) = e^{-i\nu\pi}J_\nu(z)$ and the above result (a) follows immediately; (b) can be argued in like manner.]

66. When $z = x+yi$, x and y being large enough, show that, approximately,

$$e^{\frac{1}{2}\nu\pi i}J_\nu(z)+J_{-\nu}(z) \doteqdot \frac{1}{\sqrt{(2\pi z)}}\{e^{(z-\frac{1}{4}\pi)i}(1+e^{\frac{1}{2}\nu\pi i})[\zeta_\nu(z)+i\xi_\nu(z)]\}$$

(in the fourth quadrant)

$$\doteqdot \frac{1}{\sqrt{(2\pi z)}}\{e^{-(z-\frac{1}{4}\pi+\frac{1}{2}\nu\pi)i}(1+e^{\frac{3}{2}\nu\pi i})[\zeta_\nu(z)-i\xi_\nu(z)]\}$$

(in the first quadrant).

67. The input impedance of a very long tapered loaded submarine cable is given by (see § 8, Chap. VII)

$$Z_i \doteqdot \sqrt{\left(\frac{\mathbf{Z}_0}{\mathbf{Y}_0}\right)}\{[e^{\frac{1}{3}\pi i}J_{\frac{2}{3}}(z)+J_{-\frac{2}{3}}(z)]/i[J_{\frac{1}{3}}(z)-e^{\frac{1}{3}\pi i}J_{-\frac{1}{3}}(z)]\}.$$

Using the asymptotic expansion of $J_\nu(z)$ in example 65 (a), show that, when $|z|$ is large in the second quadrant,

$$Z_i \doteqdot \sqrt{\left(\frac{\mathbf{Z}_0}{\mathbf{Y}_0}\right)}\{[\zeta_{\frac{2}{3}}(z)+i\xi_{\frac{2}{3}}(z)]/[\zeta_{\frac{1}{3}}(z)+i\xi_{\frac{1}{3}}(z)]\} \doteqdot \sqrt{\left(\frac{\mathbf{Z}_0}{\mathbf{Y}_0}\right)}\left\{\left(1+\frac{35}{432z^2}\right)+\frac{i}{6z}\right\}.$$

68. Show that, when $|z|$ is large enough [52] and $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$,

$$J_{\frac{2}{3}}(z)J_{\frac{1}{3}}(z)+J_{-\frac{2}{3}}(z)J_{-\frac{1}{3}}(z) \doteqdot \frac{\sqrt{3}}{\pi z}[\zeta_{\frac{2}{3}}(z)\zeta_{\frac{1}{3}}(z)+\xi_{\frac{2}{3}}(z)\xi_{\frac{1}{3}}(z)].$$

69. Show that, when $|z_1|$ and $|z_2|$ are large enough [52] and $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$,

$$J_{\frac{2}{3}}(z_1)J_{\frac{1}{3}}(z_2)+J_{-\frac{2}{3}}(z_1)J_{-\frac{1}{3}}(z_2)$$
$$\doteqdot \frac{1}{2\pi}\sqrt{\left(\frac{3}{z_1 z_2}\right)}\{[\zeta_{\frac{2}{3}}(z_1)-i\xi_{\frac{2}{3}}(z_1)][\zeta_{\frac{1}{3}}(z_2)+i\xi_{\frac{1}{3}}(z_2)]e^{i(z_2-z_1)}+$$
$$+[\zeta_{\frac{2}{3}}(z_1)+i\xi_{\frac{2}{3}}(z_1)][\zeta_{\frac{1}{3}}(z_2)-i\xi_{\frac{1}{3}}(z_2)]e^{i(z_1-z_2)}\}.$$

When $z_1 = z_2$ verify that the formulae in examples 68, 69 are identical.

70. Show that, when $\mu = 2$ (a conical horn), the solution of equation (77) is $\phi = \frac{B_2}{y}e^{-ikx}$. Hence show that the velocity potential at a distance r from the centre of a radially pulsating sphere of radius a is

$$\phi = U\frac{a^2}{r}\left(\frac{1-ika}{1+k^2a^2}\right)e^{-ik(r-a)},$$

where U is the surface velocity $-\left(\frac{\partial\phi}{\partial r}\right)_{r=a}$. [Put $y = r$ and $x_0 = a$.]

71. The expansion curve of a loud-speaker horn of great length is $A = B_0(x+x_0)^3$ the symbols having the same meanings as in the text. Find a formula for the velocity potential ϕ if the diaphragm velocity is v_0. If the root mean square value of v_0 is 250 cm. sec.$^{-1}$ find the sound pressure 350 cm. from the diaphragm, at a frequency of 75 ∼ ($x_0 = 200$ cm.).

72. The power delivered by the vibrating diaphragm to a Bessel horn of great length is the product of axial velocity, area, and the in-phase component of the pressure. Show that the power is

$$P = \rho_0 c A_0 v_0^2 \left|\frac{H_\nu^{(2)}(kx_0)}{H_{\nu+1}^{(2)}(kx_0)}\right| \sin(\theta_2-\theta_1),$$

where the symbols have the same meanings as in the text.

73. The equation of a Bessel loud-speaker horn is $A = B_0(x+x_0)^5$, where $x_0 = 100$ cm. Draw a vector diagram showing the phase relationship between p, v, and ϕ, 200 cm. from the diaphragm at 100 ∼.

74. The velocity potential at an axial distance x from the vertex of a horn intensifier, whose expansion curve is $A = A_0 x$, is

$$\phi = \{A_1 J_0(kx)+B_1 Y_0(kx)\}e^{i\omega t}.$$

The horn is closed by an immobile disk at a distance x_1 from the vertex, whilst at a distance $x_2 \gg x_1$ it is open to the atmosphere, i.e. the length of the horn is x_2-x_1. The acoustical impedance z_{a1} at x_1 is infinite, since all progress to the incoming wave is prevented. At the open end where $x = x_2$, the acoustical impedance is z_{a2}. If the sound pressure $p = \rho_0\dfrac{\partial\phi}{\partial t}$, the air particle velocity $v = -\dfrac{\partial\phi}{\partial x}$, and the acoustical impedance $z_a = p/Av$, where A is the cross-sectional area at x, find the ratio of the pressure at the rigid disk to that just outside the horn. The pressure at x_2 just inside the horn is p_2, whilst that outside is p_3, so (p_2-p_3) the fall in pressure is equal to $A_2 z_{a2} v_2$.

$$\left[\frac{p_1}{p_3} = \frac{[Y_0(kx_1)J_1(kx_1)-J_0(kx_1)Y_1(kx_1)]}{[Y_0(kx_2)J_1(kx_1)-J_0(kx_2)Y_1(kx_1)]+i\dfrac{A_2 k z_{a2}}{\rho_0\omega}\{Y_1(kx_2)J_1(kx_1)-J_1(kx_2)Y_1(kx_1)\}}.\right.$$

The complete solution of the differential equation is required here owing to the presence of the reflected wave.]

75. In example 74, if $A = A_0 x^m$, then $\phi = \dfrac{1}{x^n}\{A_1 J_n(kx)+B_1 Y_n(kx)\}e^{i\omega t}$ provided n is a positive integer $\left(=\dfrac{m-1}{2}\right)$. Find p_1/p_3 if the other conditions hold and $m = 3$, i.e. $n = 1$.

$$\left[\frac{p_1}{p_3} = \frac{\dfrac{x_2}{x_1}[Y_1(kx_1)-\varphi J_1(kx_1)]}{[Y_1(kx_2)-\varphi J_1(kx_2)]+\dfrac{iA_2 k z_{a2}}{\rho_0\omega}[Y_2(kx_2)-\varphi J_2(kx_2)]}\right.$$

where $\varphi = \dfrac{Y_2(kx_1)}{J_2(kx_1)}$. This is identical with the solution to 74 except that the orders of the functions are increased by unity.]

V

ADDITIONAL INTEGRALS INVOLVING BESSEL FUNCTIONS

1. Example

EVALUATE [5] $$\int_0^a J_0(kz)\left(1-\frac{z^2}{a^2}\right)^{n+1} z\,dz,$$

n being a positive integer. This integral occurs in finding the 'accession to inertia' of a flexible disk vibrating in a fluid. Substituting $z = a\sin\theta$ we obtain $\left(1-\frac{z^2}{a^2}\right) = \cos^2\theta$, $dz = a\cos\theta\,d\theta$, whilst the limits of integration are now 0 and $\frac{1}{2}\pi$. Thus we get

$$a^2\int_0^{\frac{1}{2}\pi} J_0(ka\sin\theta)\cos^{2n+3}\theta\sin\theta\,d\theta. \tag{1}$$

Expanding the Bessel function, (1) becomes

$$a^2\int_0^{\frac{1}{2}\pi}\left\{\sin\theta-\frac{y^2\sin^3\theta}{2^2}+\frac{y^4\sin^5\theta}{2^2.4^2}-\ldots\right\}\cos^{2n+3}\theta\,d\theta, \tag{2}$$

where $y = ka$. The first term in (2) is

$$\int_0^{\frac{1}{2}\pi}\sin\theta\cos^{2n+3}\theta\,d\theta = -\int_0^{\frac{1}{2}\pi}\cos^{2n+3}\theta\,d\cos\theta = 1/(2n+4).$$

By the well-known reduction formula

$$\int_0^{\frac{1}{2}\pi}\sin^{m+1}\theta\cos^{2n+3}\theta\,d\theta = \frac{m}{2n+m+4}\int_0^{\frac{1}{2}\pi}\sin^{m-1}\theta\cos^{2n+3}\theta\,d\theta,$$

provided $m > 0$.

Taking $m = 2$ we obtain

$$\int_0^{\frac{1}{2}\pi}\sin^3\theta\cos^{2n+3}\theta\,d\theta = \frac{2}{2n+6}\int_0^{\frac{1}{2}\pi}\sin\theta\cos^{2n+3}\theta\,d\theta$$
$$= \frac{2}{(2n+4)(2n+6)}.$$

For $m = 4$,

$$\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos^{2n+3}\theta\,d\theta = \frac{4.2.1}{(2n+8)(2n+6)(2n+4)}.$$

Using these evaluations we find that (2) can be written

$$\int = \frac{a^2}{2n+4}\left\{1-\frac{y^2}{2(2n+6)}+\frac{y^4}{2.4(2n+6)(2n+8)}-\ldots\right\}. \tag{3}$$

Now

$$J_{n+2}(y) = \frac{y^{n+2}}{2^{n+2}\Gamma(n+3)}\left\{1-\frac{y^2}{2(2n+6)}+\frac{y^4}{2.4(2n+6)(2n+8)}-\ldots\right\},$$

so the bracketed series in (3) is

$$\frac{2^{n+2}\Gamma(n+3)}{y^{n+2}}J_{n+2}(y). \tag{4}$$

Consequently

$$\int\limits_0^a J_0(kz)\left(1-\frac{z^2}{a^2}\right)^{n+1}z\,dz = \frac{2^{n+2}(n+2)\Gamma(n+2)a^2}{2(n+2)y^{n+2}}J_{n+2}(y),$$

or

$$a^2\int\limits_0^{\frac{1}{2}\pi} J_0(ka\sin\theta)\cos^{2n+3}\theta\sin\theta\,d\theta = 2^{n+1}\Gamma(n+2)a^2\frac{J_{n+2}(ka)}{(ka)^{n+2}}. \tag{5}$$

2. General integral

The following integral can be established [93] in like manner provided the real parts of μ and ν exceed -1 to secure convergence

$$\int\limits_0^{\frac{1}{2}\pi} J_\nu(z\sin\theta)\cos^{2\mu+1}\theta\sin^{\nu+1}\theta\,d\theta = \frac{2^\mu\Gamma(\mu+1)}{z^{\mu+1}}J_{\nu+\mu+1}(z). \tag{6}$$

Applying (6) to the preceding case, $2\mu+1 = 2n+3$, so $\mu = n+1$ and $\nu = 0$. Thus we obtain $\frac{2^{n+1}\Gamma(n+2)J_{n+2}(z)}{z^{n+2}}$, this being identical with (5) except for the factor a^2 which does not appear in (6). The integral in (5) is required in finding the 'accession to inertia' or extra mass due to the cyclical flow of fluid in the neighbourhood of a disk vibrating in an infinite plane. The general form in (6) is known as Sonine's first finite integral.

3. Example

Evaluate $$\int\limits_0^{\frac{1}{2}\pi} e^{ic\cos\theta}\cos\theta\sin\theta\,J_0(z\sin\theta)\,d\theta.$$

Expanding the exponential function, we see that the general term in the integral is

$$\frac{i^n c^n}{n!}\int\limits_0^{\frac{1}{2}\pi} J_0(z\sin\theta)\cos^{n+1}\theta\sin\theta\,d\theta.$$

Using (6) we have $\nu = 0$, $2\mu+1 = n+1$, or $\mu = \frac{1}{2}n$. Thus the general term is

$$\frac{i^n c^n}{n!} \frac{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)}{z^{\frac{1}{2}n+1}} J_{\frac{1}{2}n+1}(z). \tag{7}$$

The complete integral is found, therefore, by putting $n = 0, 1, 2,\ldots,$ in (7) and adding the various terms. The result is of the form A_1+iB_1 where

$$A_1 = \left\{\frac{J_1(z)}{z} - c^2\frac{J_2(z)}{z^2} + \frac{c^4}{1.3}\frac{J_3(z)}{z^3} - \frac{c^6}{1.3.5}\frac{J_4(z)}{z^4} + \ldots\right\} \tag{8}$$

and

$$B_1 = \sqrt{\left(\frac{\pi}{2}\right)}\left\{c\frac{J_{\frac{3}{2}}(z)}{z^{\frac{3}{2}}} - \frac{c^3}{2}\frac{J_{\frac{5}{2}}(z)}{z^{\frac{5}{2}}} + \frac{c^5}{2.4}\frac{J_{\frac{7}{2}}(z)}{z^{\frac{7}{2}}} - \frac{c^7}{2.4.6}\frac{J_{\frac{9}{2}}(z)}{z^{\frac{9}{2}}} + \ldots\right\}. \tag{9}$$

4. Infinite integral for product of two Bessel functions

In certain problems in acoustical work, and in the applications of thermionic valves, it is necessary to use integrals of the type [93]

$$\int_0^\infty \frac{J_\mu(az)J_\nu(bz)}{z^\beta}\,dz = \frac{b^\nu\Gamma\{\frac{1}{2}(\nu+\mu-\beta+1)\}}{2^\beta a^{\nu-\beta+1}\Gamma(\nu+1)\Gamma\{\frac{1}{2}(\mu-\nu+\beta+1)\}}\times$$
$$\times F[\tfrac{1}{2}(\nu-\mu-\beta+1), \tfrac{1}{2}(\nu+\mu-\beta+1); (\nu+1); b^2/a^2]. \tag{10}$$

This result only holds when the integral is convergent, a condition which is fulfilled when $R(\beta) > -1$, $R(\mu+\nu-\beta+1) > 0$, and $a > b > 0$. To establish this integral and investigate its convergence is beyond our present purpose. Convergence at the upper limit when $z \to \infty$, can be investigated by aid of the asymptotic expansion (52) in Chapter IV. (10) is known as the Weber-Schafheitlin integral.

If we put $\nu = 0$ and $\mu = \beta$, (10) becomes

$$\int_0^\infty \frac{J_\mu(az)J_0(bz)}{z^\mu}\,dz = \frac{\Gamma(\frac{1}{2})}{2^\mu a^{1-\mu}\Gamma(\mu+\frac{1}{2})} F(-\mu+\tfrac{1}{2}, \tfrac{1}{2}, 1, b^2/a^2),$$

or

$$\int_0^\infty \frac{J_\mu(az)J_0(bz)}{(az)^\mu}\,dz = \frac{\sqrt{\pi}}{2^\mu\Gamma(\mu+\frac{1}{2})}\frac{1}{a}F(-\mu+\tfrac{1}{2}, \tfrac{1}{2}, 1, b^2/a^2). \tag{11}$$

5. Example [5]

Evaluate

$$\int_0^\infty J_0(kr)\,dk \int_0^a J_0(kr)(1-\varphi r^2/a^2)r\,dr = \chi.$$

This integral occurs in determining the 'accession to inertia' or added

mass when a flexible disk vibrates in a fluid. Taking the first integral we have

$$\chi_1 = \int_0^a J_0(kr)(1-\varphi r^2/a^2)r\,dr = \int_0^a J_0(kr)r\,dr-(\varphi/a^2)\int_0^a J_0(kr)r^3\,dr$$

$$= \frac{1}{k^2}\int_0^{ka} J_0(z)z\,dz-\frac{\varphi}{k^4a^2}\int_0^{ka} J_0(z)z^3\,dz,$$

where $z = kr$.

Using § 5, Chap. III,

$$\chi_1 = \frac{1}{k^2}[zJ_1(z)]_0^{ka}-\frac{\varphi}{k^4a^2}\int_0^{ka} z^2d\{zJ_1(z)\}$$

$$= a^2\frac{J_1(ka)}{ka}-\frac{\varphi}{k^4a^2}[z^3J_1(z)-2z^2J_2(z)]_0^{ka},$$

so $$\chi_1 = a^2\left\{\frac{J_1(ka)}{ka}(1-\varphi)+2\varphi\frac{J_2(ka)}{k^2a^2}\right\}. \tag{12}$$

Thus $$\chi = a^2\int_0^\infty\left\{(1-\varphi)\frac{J_1(ka)J_0(kr)}{ka}+2\varphi\frac{J_2(ka)J_0(kr)}{(ka)^2}\right\}dk. \tag{13}$$†

Using (11) the first term of (13) becomes

$$\frac{a(1-\varphi)\sqrt{\pi}}{2\Gamma(\frac{3}{2})}F(-\tfrac{1}{2},\tfrac{1}{2},1,r^2/a^2) = a(1-\varphi)F(-\tfrac{1}{2},\tfrac{1}{2},1,r^2/a^2).$$

Similarly, the second integral in (13) yields $\frac{2}{3}a\varphi F(-\frac{3}{2},\frac{1}{2},1,r^2/a^2)$. By adding these hypergeometric functions we find that

$$\chi = a\{(1-\varphi)F_1+\tfrac{2}{3}\varphi F_2\}, \tag{14}$$

where $F_1 = F(-\frac{1}{2},\frac{1}{2},1,r^2/a^2)$, $F_2 = F(-\frac{3}{2},\frac{1}{2},1,r^2/a^2)$.

EXAMPLES

1. Evaluate $$\int_0^{\frac{1}{2}\pi} J_0(z\sin\phi)\sin\phi\cos^3\phi\,d\phi.$$ $\left[\frac{2}{z^2}J_2(z).\right]$

2. Evaluate $$\int_0^{\frac{1}{2}\pi} J_0(z\sin\phi)\sin\phi\,d\phi - \int_0^{\frac{1}{2}\pi} J_0(z\sin\phi)\sin^3\phi\,d\phi.$$ $\left[\frac{1}{z}\sqrt{\left(\frac{\pi}{2z}\right)}J_{\frac{3}{2}}(z).\right]$

3. Evaluate $$\tfrac{1}{4}\int_0^{\frac{1}{2}\pi} J_1(z\sin\phi)\sin^2 2\phi\,d\phi.$$ $\left[\frac{1}{z}\sqrt{\left(\frac{\pi}{2z}\right)}J_{\frac{5}{2}}(z).\right]$

† It should be observed that the variable is now k.

4. Evaluate $$\int_0^{\frac{1}{2}\pi} e^{ia\cos\theta}\sin^2\theta\cos^3\theta\,J_1(z\sin\theta)\,d\theta.$$

$$\left[\sum_{n=0}^{\infty}\frac{(ia)^n}{n!}\frac{2^m\Gamma(m+1)}{z^{m+1}}J_{m+2}(z),\ m=(\tfrac{1}{2}n+1).\right]$$

5. Evaluate the following integral, which occurs in finding the inertia component of the velocity potential [5] at the surface of a disk vibrating in an infinite rigid plane: $\int_0^\infty J_0(kr)\,dk\int_0^a J_0(kr)r^3\,dr$.

$$[a^3\{F(-\tfrac{1}{2},\tfrac{1}{2},1,r^2/a^2)-\tfrac{2}{3}F(-\tfrac{3}{2},\tfrac{1}{2},1,r^2/a^2)\}.]$$

6. Evaluate [5] $$\int_0^\infty J_0(kr)\,dk\int_0^a J_0(kr)(1-r^2/a^2)^{n+1}r\,dr.$$

$$\left[\frac{\pi^{\frac{1}{2}}\Gamma(n+2)}{2\Gamma(n+\frac{5}{2})}aF(-n-\tfrac{3}{2},\tfrac{1}{2},1,r^2/a^2).\quad \text{Substitute } r=a\sin\theta.\right]$$

7. Show that $$\int_0^\infty\frac{J_0(t)J_{\frac{1}{2}}(zt)\,dt}{t^{\frac{1}{2}}}=\frac{\sin^{-1}z}{\sqrt{(2\pi z)}}\quad(0<z<1).$$

[See example 2 (e), Chap. IV.]

8. The velocity potential of the fluid in contact with a membrane 10 cm. radius vibrating in a large wall at low frequencies is given by formula (14) [5] when $a = 10$ and $\varphi = 1$. Plot a curve showing the relationship between χ and r, i.e. the velocity potential distribution over the surface. Take $r = 0, 1, 2,\ldots, 10$.

9. The following integrals occur in determining modulation products in telephonic [23] work:

$$(a)\quad \int_0^\infty\frac{J_m(a\lambda)J_n(b\lambda)}{\lambda^2}\,d\lambda;\qquad (b)\quad \int_0^\infty\frac{J_m(a\lambda)J_n(b\lambda)}{\lambda^3}\,d\lambda.$$

Evaluate these integrals when $a > b > 0$; $m+n+1 > 3$.

$$\left[(a)\quad \frac{b^n\Gamma\{\frac{1}{2}(m+n-1)\}}{4a^{n-1}n!\,\Gamma\{\frac{1}{2}(m-n+3)\}}F\left(\frac{m+n-1}{2},\frac{n-m-1}{2};n+1;\frac{b^2}{a^2}\right);\right.$$

$$\left.(b)\quad \frac{b^n\Gamma\{\frac{1}{2}(m+n-2)\}}{8a^{n-2}n!\{\Gamma\frac{1}{2}(m-n+4)\}}F\left(\frac{m+n-2}{2},\frac{n-m-2}{2};n+1;\frac{b^2}{a^2}\right).\right]$$

10. Show that $\int_0^{\frac{1}{2}\pi} J_n(z\cos\theta)\cos^{n+1}\theta\,d\theta=\sqrt{\left(\frac{\pi}{2z}\right)}J_{n+\frac{1}{2}}(z)$.

[(1) Expand $J_n(z\cos\theta)$ and integrate term by term.
(2) Write $(\frac{1}{2}\pi-\theta)$ for θ in (6).]

VI

LOMMEL INTEGRALS FOR PRODUCTS OF TWO BESSEL FUNCTIONS

1. CONSIDER the differential equations

$$z^2\frac{d^2x}{dz^2}+z\frac{dx}{dz}+(l^2z^2-\nu^2)x = 0, \tag{1}$$

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+(k^2z^2-\nu^2)y = 0. \tag{2}$$

These are identical in form with Bessel's equation (11) in Chapter I, that in § 2, Chap. II just above (17a), and that of § 4, Chap. IV. Multiplying (1) by y/z and (2) by x/z we obtain

$$yz\frac{d^2x}{dz^2}+y\frac{dx}{dz}+(l^2z^2-\nu^2)\frac{xy}{z} = 0, \tag{3}$$

$$xz\frac{d^2y}{dz^2}+x\frac{dy}{dz}+(k^2z^2-\nu^2)\frac{xy}{z} = 0. \tag{4}$$

Subtracting (4) from (3)

$$yz\frac{d^2x}{dz^2}-xz\frac{d^2y}{dz^2}+y\frac{dx}{dz}-x\frac{dy}{dz} = (k^2-l^2)xyz \tag{5}$$

or

$$\frac{d}{dz}\left\{z\left(y\frac{dx}{dz}-x\frac{dy}{dz}\right)\right\} = (k^2-l^2)xyz. \tag{6}$$

Multiplying both sides of (6) by dz and integrating, we get

$$(k^2-l^2)\int^z (xy)z\,dz = z\left(y\frac{dx}{dz}-x\frac{dy}{dz}\right). \tag{7}$$

Inserting $y = J_\nu(kz)$, $x = J_\nu(lz)$ in (7) we find that when $R(\nu) > -1$ (this condition being necessary to secure convergence of the integral at the lower limit),†

$$(k^2-l^2)\int_0^z J_\nu(kz)J_\nu(lz)z\,dz = z\left\{J_\nu(kz)\frac{dJ_\nu(lz)}{dz}-J_\nu(lz)\frac{dJ_\nu(kz)}{dz}\right\}. \tag{8}$$

Now $\dfrac{dJ_\nu(lz)}{dz} = l\dfrac{dJ_\nu(lz)}{d(lz)} = lJ_\nu'(lz)$, and $\dfrac{dJ_\nu(kz)}{dz} = kJ_\nu'(kz)$, so on sub-

† The first term in the series representing the integrand is of order $2\nu+1$. But $\int z^{2\nu+1}\,dz = \dfrac{z^{2\nu+2}}{2\nu+2}$ and this only converges when $z \to 0$ if $R(2\nu+2) > 0$ or $R(\nu) > -1$, unless $\nu = n$ an integer, when $J_{-n}(z) = (-1)^nJ_n(z)$, and the integral converges for all values of n.

stitution in (8) we obtain

$$\int_0^z J_\nu(kz)J_\nu(lz)z\,dz = \frac{z}{k^2-l^2}\{lJ_\nu(kz)J_\nu'(lz)-kJ_\nu(lz)J_\nu'(kz)\}. \tag{9}$$

Since $J_\nu'(z)$ is not tabulated, (9) can be put in a form amenable to numerical calculation by aid of recurrence relationships. Thus from (19), Chapter II, writing ν for n and altering the argument accordingly,

$$J_\nu'(kz) = \frac{\nu}{kz}J_\nu(kz)-J_{\nu+1}(kz),$$

and

$$J_\nu'(lz) = \frac{\nu}{lz}J_\nu(lz)-J_{\nu+1}(lz).$$

Substituting these values of $J_\nu'(kz)$ and $J_\nu'(lz)$ in the right-hand side of (9) we obtain, with $R(\nu) > -1$,

$$\int_0^z J_\nu(kz)J_\nu(lz)z\,dz = \frac{z}{k^2-l^2}\{kJ_\nu(lz)J_{\nu+1}(kz)-lJ_\nu(kz)J_{\nu+1}(lz)\}. \tag{10}$$

Again, from (21), Chapter II, we have $J_\nu'(kz) = -\frac{\nu}{kz}J_\nu(kz)+J_{\nu-1}(kz)$ with a similar expression for $J_\nu'(lz)$. On substitution in (9) we get, when $R(\nu) > -1$,

$$\int_0^z J_\nu(kz)J_\nu(lz)z\,dz = \frac{z}{k^2-l^2}\{lJ_{\nu-1}(lz)J_\nu(kz)-kJ_{\nu-1}(kz)J_\nu(lz)\}. \tag{11}$$

From (9), (10), and (11) we see that the bracketed quantities are identical.

In particular if in (10) we put $\nu = 0$,

$$\int_0^a J_0(kz)J_0(lz)z\,dz = \frac{a}{k^2-l^2}\{kJ_0(la)J_1(ka)-lJ_0(ka)J_1(la)\}. \tag{12}$$

This formula is useful in problems associated with the sound radiation from membranes and vibrating flexible circular disks. The above expressions for integral products were first given by the German mathematician E. C. J. von Lommel.

2. Example

Evaluate
$$\Theta = \int_0^a\int_0^{2\pi} xJ_0(k_1 x)e^{ikx\sin\phi\cos\theta}\,dxd\theta.$$

The integral can be written $\int_0^a xJ_0(k_1 x)\,dx\int_0^{2\pi} e^{ikx\sin\phi\cos\theta}\,d\theta$. Writing

$z = kx\sin\phi$ in the exponential of the first integral and using (19), Chap. III, its value is $2\pi J_0(kx\sin\phi)$. Thus

$$\Theta = 2\pi\int\limits_0^a J_0(kx\sin\phi)J_0(k_1 x)x\,dx.$$

Using (10) and writing $l = k\sin\phi$, $k = k_1$ we obtain

$$\Theta = \frac{2\pi a}{k_1^2-k^2\sin^2\phi}\{k_1 J_0(ka\sin\phi)J_1(k_1 a)-k\sin\phi\, J_0(k_1 a)J_1(ka\sin\phi)\}. \tag{13}$$

At a vibrational mode of a circular membrane it was shown in Chapter I that $J_0(k_1 a) = 0$. Under this condition

$$\Theta = \left[\frac{2\pi}{k_1^2-k^2\sin^2\phi}\right]k_1 aJ_1(k_1 a)J_0(ka\sin\phi). \tag{14}$$

This example pertains to the distribution of sound radiation from a circular membrane vibrating in a rigid plane of infinite extent [9, 16].

3. Special case

When $k = l$, the result in (9) assumes the indeterminate form 0/0. By using Taylor's theorem it can be shown that when $R(\nu) > -1$

$$\int\limits_0^z z[J_\nu(kz)]^2\,dz = \tfrac{1}{2}z^2\left[\{J_\nu'(kz)\}^2+\left(1-\frac{\nu^2}{k^2z^2}\right)\{J_\nu(kz)\}^2\right]. \tag{15}$$

In particular when $\nu = 0$, (15) reduces, with the aid of (20), Chap. II, to

$$\int\limits_0^z zJ_0^2(kz)\,dz = \tfrac{1}{2}z^2[J_1^2(kz)+J_0^2(kz)]. \tag{16}$$

4. Integrals for any two cylinder functions

By a procedure similar to that used in (1) to (9) the following general integrals can be established.

$$\int^z \mathfrak{C}_\nu(kz)\overline{\mathfrak{C}}_\nu(lz)z\,dz = \frac{z}{k^2-l^2}\{k\overline{\mathfrak{C}}_\nu(lz)\mathfrak{C}_{\nu+1}(kz)-l\mathfrak{C}_\nu(kz)\overline{\mathfrak{C}}_{\nu+1}(lz)\}, \tag{17}$$

the bar signifying that the two cylinder functions are different. When $k = l$, (17) is indeterminate, and in this case

$$\int^z \mathfrak{C}_\nu(kz)\overline{\mathfrak{C}}_\nu(kz)z\,dz$$
$$= \tfrac{1}{4}z^2\{2\mathfrak{C}_\nu(kz)\overline{\mathfrak{C}}_\nu(kz)-\mathfrak{C}_{\nu-1}(kz)\overline{\mathfrak{C}}_{\nu+1}(kz)-\mathfrak{C}_{\nu+1}(kz)\overline{\mathfrak{C}}_{\nu-1}(kz)\}. \tag{18}$$

If $\mathfrak{C}_\nu = \overline{\mathfrak{C}}_\nu$, (18) becomes

$$\int^z z\mathfrak{C}_\nu^2(kz)\,dz = \tfrac{1}{2}z^2\{\mathfrak{C}_\nu^2(kz)-\mathfrak{C}_{\nu-1}(kz)\mathfrak{C}_{\nu+1}(kz)\} \tag{19}$$

$$= \tfrac{1}{2}z^2\left\{\left(1-\frac{\nu^2}{k^2z^2}\right)\mathfrak{C}_\nu^2(kz)+\mathfrak{C}_\nu'^2(kz)\right\}. \tag{20}$$

These results are due to Lommel.

5. Example [6]

Evaluate
$$\int_0^a Y_0(k_1 x)J_0(kx\sin\phi)x\,dx.$$

Using (17) we have $\nu = 0$, $\mathfrak{C}_0 = Y_0$, $\overline{\mathfrak{C}}_0 = J_0$, $k = k_1$, $l = k\sin\phi$, so we obtain

$$\left[\frac{x}{k_1^2-k^2\sin^2\phi}\{k_1 J_0(kx\sin\phi)Y_1(k_1 x)-k\sin\phi\, Y_0(k_1 x)J_1(kx\sin\phi)\}\right]_0^a.$$

From (3) and (5), Chap. II ($x\to 0$), the lower limit gives $\dfrac{2}{\pi(k_1^2-k^2\sin^2\phi)}$, so

$$\int_0^a Y_0(k_1 x)J_0(kx\sin\phi)x\,dx - \frac{2}{\pi(k_1^2-k^2\sin^2\phi)}$$

$$= \frac{a}{k_1^2-k^2\sin^2\phi}\{k_1 J_0(ka\sin\phi)Y_1(k_1 a)-k\sin\phi\, Y_0(k_1 a)J_1(ka\sin\phi)\}. \tag{21}$$

6. Formulae for squares and products of Bessel functions

In certain acoustical problems pertaining to the power radiated from vibrating disks, integrals of the type $\displaystyle\int \frac{J_n^2(kz)}{z^p}\,dz$, $\displaystyle\int \frac{J_n(kz)J_m(kz)}{z^p}\,dz$ are encountered. These can be evaluated in the form of an infinite series. By multiplying together the series for the Bessel functions in the integrands the following results are obtained:

$$J_n^2(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(2n+2r)!(\frac{1}{2}z)^{2n+2r}}{r!(2n+r)!\{(n+r)!\}^2}. \tag{22}$$

$$J_\nu^2(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(2\nu+2r)_r(\frac{1}{2}z)^{2\nu+2r}}{r!\{\Gamma(\nu+r+1)\}^2}, \tag{22a}$$

where $(2\nu+2r)_r = (2\nu+2r)(2\nu+2r-1)...(2\nu+r+1)$.

$$J_n(z)J_m(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(n+m+2r)!(\frac{1}{2}z)^{n+m+2r}}{r!(n+m+r)!(n+r)!(m+r)!}. \tag{23}$$

$$J_\mu(z)J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(\mu+\nu+2r)_r(\frac{1}{2}z)^{\mu+\nu+2r}}{r!\,\Gamma(\mu+r+1)\Gamma(\nu+r+1)}. \tag{23 a}$$

$$J_\mu(az)J_\nu(bz) = \frac{(\frac{1}{2}az)^\mu(\frac{1}{2}bz)^\nu}{\Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}az)^{2r}F(-r,\,-\mu-r;\,\nu+1;\,b^2/a^2)}{r!\,\Gamma(\mu+r+1)}, \tag{24}$$

where F denotes the hypergeometric function defined in Chapter IV.

7. Example [7]

Evaluate
$$\int_0^{\frac{1}{2}\pi} \frac{J_1^2(ka\sin\phi)}{(ka\sin\phi)^2}\sin\phi\,d\phi,$$
and show that
$$2\int_0^{\frac{1}{2}\pi} \frac{J_1^2(z\sin\phi)}{\sin\phi}\,d\phi = \left[1-\frac{J_1(2z)}{z}\right].$$

From (22) the general term in the expansion of $\dfrac{J_1^2(ka\sin\phi)}{(ka\sin\phi)^2}$ is
$$\frac{(-1)^r(2r+2)!}{2^{2r+2}r!\,(r+2)!\,[(r+1)!]^2}(ka\sin\phi)^{2r} = \Pi(ka\sin\phi)^{2r}, \tag{25}$$
where Π represents the fraction on the left. Using the sine reduction formula we obtain
$$\Pi\int_0^{\frac{1}{2}\pi} (ka\sin\phi)^{2r}\sin\phi\,d\phi = \Pi(ka)^{2r}\int_0^{\frac{1}{2}\pi}\sin^{2r+1}\phi\,d\phi = \Pi\,\frac{2^r r!\,(ka)^{2r}}{1.3.5...(2r+1)}. \tag{26}$$
Now $(2r+2)!$ in $(25) = (2r+2)(2r+1)2r...3.2.1$
$$= 2^{r+1}(r+1)!\,(2r+1)(2r-1)...5.3.1.$$
Using this in (26), we obtain the general term in the expansion of the original integral, i.e. $\dfrac{(-1)^r(ka)^{2r}}{2(r+1)!\,(r+2)!}$.

Thus
$$\int_0^{\frac{1}{2}\pi} \frac{J_1^2(ka\sin\phi)}{(ka\sin\phi)^2}\sin\phi\,d\phi = \frac{1}{2}\sum_{r=0}^{\infty}\frac{(-1)^r(ka)^{2r}}{(r+1)!\,(r+2)!} \tag{27}$$
$$= \frac{1}{2}\left\{\frac{1}{1!\,2!}-\frac{(ka)^2}{2!\,3!}+\frac{(ka)^4}{3!\,4!}-...\right\}. \tag{28}$$
In (27) write $ka = z$ and multiply both sides by $2z^2$, then
$$2\int_0^{\frac{1}{2}\pi} \frac{J_1^2(z\sin\phi)}{\sin\phi}\,d\phi = \sum_{r=0}^{\infty}\frac{(-1)^r z^{2r+2}}{(r+1)!\,(r+2)!}. \tag{29}$$

But $$J_1(2z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r!(r+1)!} = z - \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+3}}{(r+1)!(r+2)!}. \tag{30}$$

Thus $$\sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+2}}{(r+1)!(r+2)!} = \left[1 - \frac{J_1(2z)}{z}\right], \tag{31}$$

so the identity is established. This latter piece of analysis occurs in evaluating the power radiated as sound by a rigid disk vibrating in an infinite rigid plane [83].

8. Example [7, 83]

Show that $$\Psi = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r}}{(r!)^2(2r+1)} = \frac{1}{z} \sum_{r=0}^{\infty} J_{2r+1}(2z).$$

We have $$\frac{d}{dz}(z\Psi) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r}}{(r!)^2} = J_0(2z),$$

and by integration
$$\Psi = \frac{1}{z} \int_0^z J_0(2z)\, dz$$
$$= \frac{1}{z} \sum_{r=0}^{\infty} J_{2r+1}(2z) \tag{32}$$
from (41), Chapter III.

EXAMPLES

1. Plot $J_0(z)J_0(2z)z$ from $z = 0$ to 5 and find the area between the curve and the z-axis using Simpson's rule (or an alternative). Check by integration. [$\frac{5}{3}[J_0(10)J_1(5) - 2J_0(5)J_1(10)] = 0{\cdot}16$.]

2. Evaluate $\int_0^a J_0(kr\sin\phi)J_0(lr\cos\phi)r\, dr$ if $k\sin\phi \neq l\cos\phi$.

$$\left[\frac{a}{k^2\sin^2\phi - l^2\cos^2\phi}\{k\sin\phi\, J_0(la\cos\phi)J_1(ka\sin\phi) - -l\cos\phi\, J_0(ka\sin\phi)J_1(la\cos\phi)\}.\right]$$

3. Evaluate [6] $\int_0^a J_0(k_1 x)Y_0(k_1 x)x\, dx.$
[$\frac{1}{2}a^2\{Y_0(k_1 a)J_0(k_1 a) + Y_1(k_1 a)J_1(k_1 a)\}$.]

4. Evaluate [6] $\int_b^a J_0(kr\sin\phi)Y_0(k_1 r)r\, dr,\ k_1 \neq k\sin\phi.$

$$\left[\frac{1}{k_1^2 - k^2\sin^2\phi}\{k_1[aJ_0(ka\sin\phi)Y_1(k_1 a) - bJ_0(kb\sin\phi)Y_1(k_1 b)] -k\sin\phi[aY_0(k_1 a)J_1(ka\sin\phi) - bY_0(k_1 b)J_1(kb\sin\phi)]\}.\right]$$

5. Evaluate $$\int_0^a J_0^2(k_1 r) r\,dr. \qquad [\tfrac{1}{2}a^2[J_1^2(k_1 a)+J_0^2(k_1 a).]]$$

[Integrals of the type in examples 2 to 5 occur in determining the distribution of sound from disks and membranes vibrating in an infinite plane.]

6. Evaluate [7] $$\int_0^{\frac{1}{2}\pi} \frac{J_2^2(x)}{x^4}\sin\phi\,d\phi, \quad \text{where } x = ka\sin\phi,\ z = ka.$$

$$\left[\frac{1}{4}\sum_{r=0}^{\infty}\frac{(-1)^r(2r+3)}{(r+2)!\,(r+4)!}(ka)^{2r} = \frac{1}{4z^6}\left\{\frac{z^2}{2!}+\frac{z^4}{3!}-J_2(2z)-2z\,J_3(2z)\right\}.\right]$$

7. Evaluate [7]

$$\int_0^{\frac{1}{2}\pi}\left\{\frac{J_1^2(x)}{x^2}-\frac{8J_1(x)J_2(x)}{x^3}+\frac{16J_2^2(x)}{x^4}\right\}\sin\phi\,d\phi, \quad \text{where } x = ka\sin\phi.$$

$$\left[\frac{1}{4}\sum_{r=0}^{\infty}\frac{(-1)^r r(r-1)(ka)^{2r}}{(r+1)!\,(r+4)!} = \frac{1}{4}\left\{\frac{1}{3(ka)^2}+\frac{J_3(2ka)}{(ka)^3}\left[1-\frac{2}{(ka)^2}\right]-\frac{6J_4(2ka)}{(ka)^4}\right\}.\right]$$

8. Show that

$$[m\,J_\nu(nz)J_{\nu+1}(mz)-n\,J_\nu(mz)J_{\nu+1}(nz)] = [n\,J_{\nu-1}(nz)J_\nu(mz)-m\,J_{\nu-1}(mz)J_\nu(nz)].$$

9. Establish the identity

$$\int_0^{\frac{1}{2}\pi}\frac{J_0'^2\{z\cos(\phi-\frac{1}{2}\pi)\}}{\cos(\phi-\frac{1}{2}\pi)}\,d\phi = \frac{1}{2}\left[1+\frac{J_0'(2z)}{z}\right].$$

10. Evaluate $$\int^z J_\nu(kz)Y_\nu(lz)z\,dz.$$

$$\left[\frac{z}{k^2-l^2}\{l\,J_\nu(kz)Y_\nu'(lz)-k\,Y_\nu(kz)J_\nu'(lz)\};\ \frac{z}{k^2-l^2}\{k\,Y_\nu(lz)J_{\nu+1}(kz)-l\,J_\nu(kz)Y_{\nu+1}(lz)\}.\right]$$

11. Evaluate $$\int^z Y_\nu^2(kz)z\,dz.$$

$$\left[\tfrac{1}{2}z^2\{Y_\nu^2(kz)-Y_{\nu-1}(kz)Y_{\nu+1}(kz)\};\ \tfrac{1}{2}z^2\left\{Y_\nu'^2(kz)+\left(1-\frac{\nu^2}{k^2z^2}\right)Y_\nu^2(kz)\right\}.\right]$$

12. Evaluate $$\int^z H_\nu^{(1)}(z)H_\nu^{(2)}(z)z\,dz.$$

$$[\tfrac{1}{4}z^2\{2H_\nu^{(1)}(z)H_\nu^{(2)}(z)-H_{\nu-1}^{(1)}(z)H_{\nu+1}^{(2)}(z)-H_{\nu+1}^{(1)}(z)H_{\nu-1}^{(2)}(z)\}.]$$

13. Verify that when k_1 and k_2 are different roots of either $J_0(k_n a) = 0$ or $J_1(k_n a) = 0$,

$$\int_0^a z\,J_0(k_1 z)J_0(k_2 z)\,dz = 0.$$

14. If $f(z) = A_1 J_0(k_1 z)+A_2 J_0(k_2 z)+\ldots$, where $k_1\ldots k_n$ are the roots of $J_0(k_n z) = 0$, find the value of A_n.

$\left[A_n = \dfrac{2}{a^2 J_1^2(k_n a)}\displaystyle\int_0^a z f(z)J_0(k_n z)\,dz.\right.$ Multiply both sides by $z\,J_0(k_n z)$ and integrate from 0 to a.]

15. Show that [7] $$\sum_{r=0}^{\infty} \frac{(-1)^r z^{2r}}{(r+1)!(r+4)!} = \frac{1}{z^2 3!} - \frac{J_3(2z)}{z^5}.$$

16. Show that [7]
$$\sum_{r=0}^{\infty} \frac{(-1)^r z^{2r}}{(r+2)!(r+4)!} = \frac{1}{z^6}\left[J_2(2z) - \frac{z^2}{2!} + \frac{z^4}{3!}\right].$$

17. Show that [7, 83]
$$\sum_{m=0}^{\infty} \frac{(-1)^m z^{2n+2m}}{m!(2n+m+2)!(n+m+1)} = \left[J_{2n+2}(2z) + 2\sum_{r=0}^{\infty} J_{2n+2r+3}(2z)\right] \Big/ (n+1)z^2.$$

[See worked example § 8.]

18. Show that
$$\sum_{r=0}^{\infty} \frac{(-1)^r z^{2r}}{(r+m)!(r+n)!} = \frac{(-1)^m}{z^{m+n}}\left[J_{n-m}(2z) - \sum_{r=0}^{m-1} \frac{(-1)^r z^{2r+n-m}}{r!(r+n-m)!}\right].$$

[Formulae of the type given in examples 15 to 18 occur in determining the power radiated by flexible disks vibrating in an infinite rigid plane.]

19. Verify that when $z \to 0$ the limiting values of the following are

(*a*) $zY_0(z) = 0$; (*b*) $zY_1(z) = -2/\pi$; (*c*) $zJ_1(z)Y_0(z) = 0$;

(*d*) $zJ_0(z)Y_1(z) = -2/\pi$.

20. Verify the following limiting values when $z \to 0$ and $R(\nu) > -1$,

(*a*) $kzJ_{\nu+1}(kz)Y_\nu(lz) = 0$; (*b*) $lzJ_\nu(kz)Y_{\nu+1}(lz) = -(2/\pi)(k/l)^\nu$;

(*c*) $kzJ_\nu'(kz)Y_\nu(lz) - lzJ_\nu(kz)Y_\nu'(lz) = -(2/\pi)(k/l)^\nu$.

21. Verify the following limiting values when $z \to 0$ and $R(\nu) > 0$,

(*a*) $kzJ_\nu'(kz)Y_\nu(lz) = -(1/\pi)(k/l)^\nu$; (*b*) $lzJ_\nu(kz)Y_\nu'(lz) = (1/\pi)(k/l)^\nu$;

(*c*) that (*a*) vanishes whilst (*b*) is $2/\pi$ when $\nu = 0$.

[The results in examples 19, 20, 21, are useful in evaluating Lommel integrals at the lower limit $z = 0$. In the case of $\int_0^a J_\nu(kz)Y_\nu(lz)z\,dz$ the lower limit yields a constant term as in § 5. If $Y_\nu(lz)$ is replaced by $J_\nu(lz)$, the integral vanishes at the lower limit. See also examples 16, 17, p. 115. When $k = l$ the procedure outlined in example 43, p. 154 is used. It is well to visualize a Lommel integral as an area of integration. The value at $z = 0$ is then seen to depend upon the forms of the functions in this neighbourhood.]

VII

THE MODIFIED BESSEL FUNCTIONS $I_\nu(z)$ AND $K_\nu(z)$

1. The differential equation for $I_\nu(z)$ and $K_\nu(z)$

IN Chapter II it is shown that the Bessel functions $J_\nu(z)$, $Y_\nu(z)$, $J_{-\nu}(z)$ are solutions of the differential equation $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+\left(1-\dfrac{\nu^2}{z^2}\right)y = 0.$

If z is replaced by zi, we must write $i^{-1}\dfrac{dy}{dz}$ for $\dfrac{dy}{dz}$ and $-\dfrac{d^2y}{dz^2}$ for $\dfrac{d^2y}{dz^2}$.

Making these substitutions the above equation becomes

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\left(1+\frac{\nu^2}{z^2}\right)y = 0. \tag{1}$$

The solution of (1) is

$$y = A_1 J_\nu(zi)+B_1 J_{-\nu}(zi), \tag{2}$$

when ν is non-integral, and

$$y = A_1 J_n(zi)+B_1 Y_n(zi), \tag{2a}$$

when $\nu = n$ an integer.

It is often desirable in applications to present the solution in real instead of in imaginary form, so that the Bessel functions must be modified. Since $i^{-\nu} = e^{-\frac{1}{2}\nu\pi i}$ is constant for a given order ν, $i^{-\nu}J_\nu(zi)$ must be a solution of (1). When z is real we write $I_\nu(z) = i^{-\nu}J_\nu(zi)$, this being a modified Bessel function of the first kind of order ν. It is easy to show from (16a), Chap. IV, that

$$I_\nu(z) = \sum_{r=0}^{\infty}\frac{(\frac{1}{2}z)^{\nu+2r}}{r!\Gamma(\nu+r+1)}. \tag{3}$$

When $\nu = n$, we have

$$I_n(z) = i^{-n}J_n(zi).$$

Writing $-n$ for n we get from (17a), Chap. IV,

$$I_{-n}(z) = i^nJ_{-n}(zi) = i^n(-1)^nJ_n(zi)$$
$$= i^{-n}J_n(zi);$$

so

$$I_{-n}(z) = I_n(z). \tag{4}$$

This relationship can also be established by aid of (3). We also see from (3) that

$$I_n(-z) = (-1)^nI_n(z). \tag{4a}$$

Since

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^\pi e^{\pm iz\cos\theta}\sin^{2\nu}\theta\,d\theta; \text{ when } R(\nu) > -\tfrac{1}{2}$$

[(30), Chap. IV], we find on substituting zi for z that

$$I_\nu(z) = i^{-\nu}J_\nu(zi) = \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int\limits_0^\pi e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta. \tag{5}$$

Also $$I_\nu(z) = \tfrac{1}{2}i^{-\nu}\{H_\nu^{(1)}(zi)+H_\nu^{(2)}(zi)\}. \tag{5a}$$

The second solution of (1) is defined to be

$$K_\nu(z) = \frac{\frac{1}{2}\pi\{I_{-\nu}(z)-I_\nu(z)\}}{\sin\nu\pi} = \tfrac{1}{2}\pi i^{\nu+1}H_\nu^{(1)}(zi). \tag{6}$$

$K_\nu(z)$ is known as a modified Bessel function of the second kind of order ν. Since $I_{-\nu}(z) = I_\nu(z)$ and $\sin\nu\pi = 0$ when $\nu = n$, an integer, $K_n(z)$ is defined as the limiting value of the fraction in (6) when $\nu \to n$ (see (112), p. 165).

As in § 5, Chap. I, the condition to be fulfilled by the two solutions of (1) is that $W\{y_1, y_2\} \neq 0$. The solution of (1) can be written in various forms thus:

$$y = A_1 I_\nu(z)+B_1 K_\nu(z) \quad \text{always} \tag{7}$$

$$y = A_1 I_\nu(z)+B_1 I_{-\nu}(z) \quad \nu \text{ non-integral} \tag{8}$$

$$y = A_1 I_n(z)+B_1 K_n(z) \quad \nu = n \text{ an integer.} \tag{9}$$

It can be shown that

$$K_{-\nu}(z) = K_\nu(z) = \tfrac{1}{2}\pi i^{\nu+1}H_\nu^{(1)}(zi). \tag{9a}$$

When the differential equation is

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\left(k^2+\frac{\nu^2}{z^2}\right)y = 0,$$

the solution takes one of the forms (7), (8), (9) in which kz is written for z. The recurrence formulae and the asymptotic expansions for $I_\nu(z)$ and $K_\nu(z)$ are given in the list on pp. 163, 165, whilst both functions of order zero are plotted in Fig. 16.

Having introduced the principal Bessel functions, we can give various pairs which constitute fundamental systems of solutions as follows: (1) $J_n(z)$, $Y_n(z)$; (2) $I_n(z)$, $K_n(z)$; (3) $H_n^{(1)}(z)$, $H_n^{(2)}(z)$; (4) $J_\nu(z)$, $Y_\nu(z)$; (5) $J_\nu(z)$, $J_{-\nu}(z)$; (6) $I_\nu(z)$, $K_\nu(z)$; (7) $I_\nu(z)$, $I_{-\nu}(z)$; (8) $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$.

2. Equation whose solution contains four Bessel functions

In determining the amplitude of a centrally driven flexible disk vibrating symmetrically *in vacuo*, without inherent loss, the following equation occurs [72]:

$$\frac{d^4y}{dz^4}+\frac{2}{z}\frac{d^3y}{dz^3}-\frac{1}{z^2}\frac{d^2y}{dz^2}+\frac{1}{z^3}\frac{dy}{dz}-y = 0. \tag{10}$$

Now the expression

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}-1\right)\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1\right)y$$

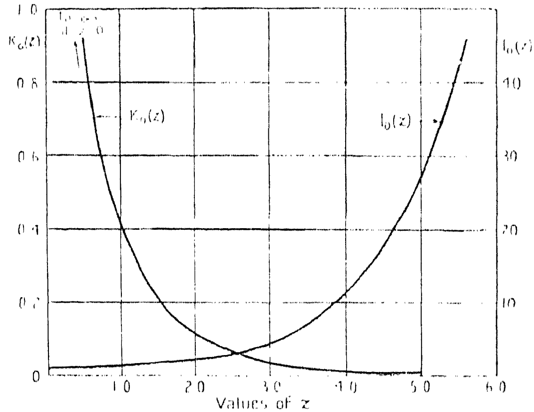


FIG. 16. The modified Bessel functions $I_0(z)$ and $K_0(z)$. $K_0(z)$ is asymptotic to both axes; the functions are not oscillatory.

is equal to the left-hand side of (10). Consequently the equation can be written

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}-1\right)\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1\right)y = 0, \tag{11}$$

or

$$(\Theta^2-1)(\Theta^2+1)y = 0, \tag{11a}$$

where†

$$\Theta^2 = \frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}.$$

If a function y satisfies

$$\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1\Big)y = 0 \qquad (1$$

† Θ^2 is an operator, not a multiplier.

and
$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}-1\right)y = 0, \tag{13}$$
it must also satisfy (10) and (11).

Now the solutions of (12) and (13) are, respectively,
$$y_1 = A_1 J_0(z)+B_1 Y_0(z)$$
and
$$y_2 = C_1 I_0(z)+D_1 K_0(z).$$
Thus the complete solution of (10), which contains four arbitrary constants, is
$$y = y_1+y_2 = A_1 J_0(z)+B_1 Y_0(z)+C_1 I_0(z)+D_1 K_0(z). \tag{14}$$
When the last term in (10) is replaced by k^4y, the argument in (14) is kz. Expression (14) gives the amplitude of a disk driven harmonically *in vacuo*. The constants A_1, B_1, C_1, D_1 are determined to comply with the conditions at the edge and at the attachment of the central driving mechanism. The drive may be on a tiny circle at the centre, or on a relatively large one, in which case the disk is to be regarded as an annulus. The same differential equation is applicable in both cases.

3. Example

If the relationship between grid voltage E and current I of a rectifying valve is $I = A_1 e^{bE}$, find the mean and root mean square values of the current when a sine wave voltage $E = E_0 \cos\omega t$ is applied across the grid and cathode.

The current $I = A_1 e^{bE_0\cos\theta}$, where $\theta = \omega t$, so its mean value over a complete cycle is
$$I_{\text{mean}} = \frac{\omega}{2\pi}\int\limits_0^{2\pi/\omega} I\,dt = \frac{A_1}{\pi}\int\limits_0^{\pi} e^{bE_0\cos\theta}\,d\theta. \tag{15}$$
Putting $\nu = 0$ in (5) and remembering that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have
$$I_0(z) = \frac{1}{\pi}\int\limits_0^{\pi} e^{\pm z\cos\theta}\,d\theta,$$
so that
$$I_{\text{mean}} = A_1 I_0(bE_0). \tag{16}$$
Also
$$I^2_{\text{r.m.s.}} = \frac{\omega}{2\pi}\int\limits_0^{2\pi/\omega} I^2\,dt = \frac{A_1^2}{\pi}\int\limits_0^{\pi} e^{2bE_0\cos\theta}\,d\theta$$
$$= A_1^2 I_0(2bE_0).$$
Hence
$$I_{\text{r.m.s.}} = A_1 I_0^{\frac{1}{2}}(2bE_0). \tag{16 a}$$

4. Example

Prove that
$$e^{z\cos\theta} = I_0(z)+2\sum_{n=1}^{\infty} I_n(z)\cos n\theta.$$
From (13), Chap. III,
$$e^{iz\cos\theta} = J_0(z)+2\sum_{n=1}^{\infty} i^n J_n(z)\cos n\theta.$$

Writing $-iz$ for z throughout we obtain
$$e^{z\cos\theta} = J_0(-iz)+2\sum_{n=1}^{\infty} i^n J_n(-iz)\cos n\theta. \tag{17}$$
From (4 a), Chap. II, $\quad J_n(-iz) = (-1)^n J_n(zi)$
$$= i^{3n} I_n(z).$$
On substituting this value of $J_n(-iz)$ in (17) the above expression for $e^{z\cos\theta}$ is obtained.

5. Electrical transmission lines

The solution of the equations for a uniform line† involves the hyperbolic functions sinh Px, cosh Px, etc. where $P = \{(\mathbf{R}+i\omega\mathbf{L})(\mathbf{G}+i\omega\mathbf{C})\}^{\frac{1}{2}}$ is the propagation coefficient of the line, and x is the distance from the sending end. As shown in § 6, Chap. IV, these functions are particular cases of Bessel functions. In modern practice it is customary to 'load' cables by wrapping the central copper conductor with very thin magnetic material of high initial permeability, of the order 4,000, thereby increasing the inductance to many times its original value. The sending current at the 'head end' of the line is usually large enough to increase the magnetic flux in the loading material to such an extent that the permeability is altered appreciably. With a sine wave current of large amplitude, the loading material may approach magnetic saturation, so the effective inductance and effective resistance vary throughout a cycle. Thus in the head end of the cable the value of P changes throughout a cycle, signals are distorted and alien frequencies are created. In duplex working, i.e. simultaneous transmission and reception, which is the most economical procedure, the receiver is connected across a balanced bridge. The cable also goes to one corner of the bridge whilst an artificial line is joined to the other corner. It is necessary to have a good balance so that the receiver is immune from serious interference due to the transmitted

† A line whose inductance **L**, capacity **C**, resistance **R**, and leakance **G**, all taken on unit length, are constant.

currents. This condition can only be obtained if the electrical characteristics of the cable are matched by those of the artificial line. This is a fairly simple matter in the case of an unloaded cable, but an accurate balance cannot be obtained with a heavily loaded cable at

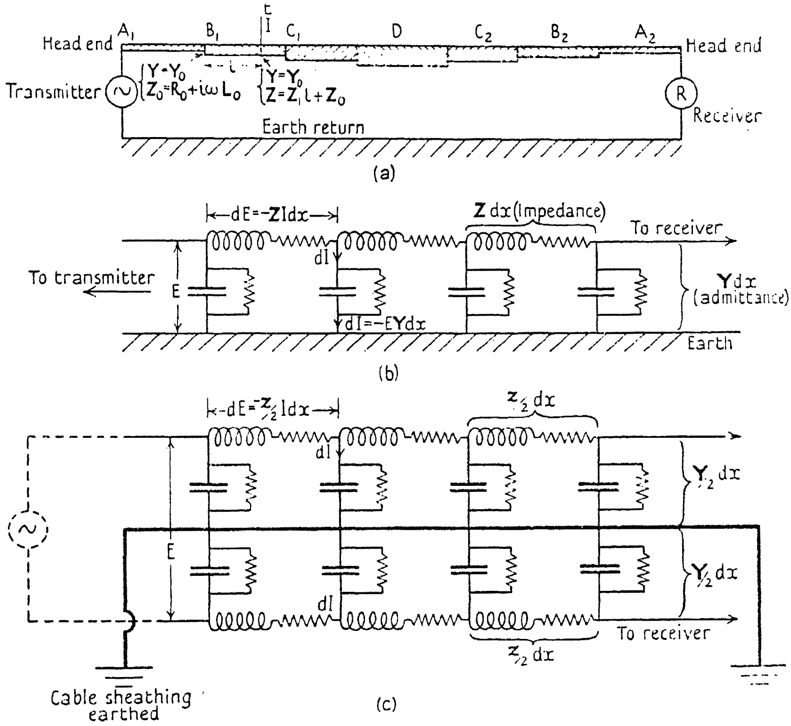


FIG. 17. (*a*) Schematic simplified diagram of electrical transmission line with graded loading and earth return. A_1 = unloaded, B_1, B_2 = linearly tapered loading, C_1, C_2 = heavier linearly tapered loading, D = constant loading $> C_1, C_2$. (*b*) Series impedance **Z** and shunt admittance **Y**. (*c*) Schematic diagram of twin-core cable where the combined go and return series impedance **Z** and shunt admittance **Y** per unit length are equal to those in (*b*). The analysis is identical for (*b*) and (*c*).

the head end of the line. To obviate this difficulty both ends of the line are either unloaded for a certain distance or loaded with special material of low permeability which does not vary with the sending current. To avoid an abrupt change in **L** and **R**, the loading is graded (see Fig. 17) and increases with the distance from the transmitter. In lines of this type the solution of the differential equations usually involves Bessel functions.

Let I and E represent, respectively, the current (sinusoidal) and potential difference at any point distant x from an origin (sometimes complex) which will be defined later, whilst $\mathbf{Z}$ is the variable series impedance per unit length due to resistance and inductance (Fig. 17). Then the impedance of a length dx is $\mathbf{Z}\,dx$, and the corresponding potential difference

$$-dE = I\mathbf{Z}\,dx$$

or

$$-\frac{dE}{dx} = I\mathbf{Z}, \tag{18}$$

the negative sign signifying that E decreases with increase in x.

If $\mathbf{Y}$ is the variable shunt admittance per unit length due to capacity and leakance, that due to a length dx is $\mathbf{Y}\,dx$. The corresponding current is

$$-dI = E\mathbf{Y}\,dx$$

or

$$-\frac{dI}{dx} = E\mathbf{Y}. \tag{19}$$

Substituting $I = -\frac{1}{\mathbf{Z}}\frac{dE}{dx}$ from (18) in (19), we have

$$\frac{d}{dx}\left\{\frac{1}{\mathbf{Z}}\frac{dE}{dx}\right\} - \mathbf{Y}E = 0. \tag{20}$$

Performing the differentiation indicated in (20), we obtain

$$\frac{1}{\mathbf{Z}}\frac{d^2E}{dx^2} - \frac{\mathbf{Z}'}{\mathbf{Z}^2}\frac{dE}{dx} - \mathbf{Y}E = 0$$

or

$$\frac{d^2E}{dx^2} - \frac{\mathbf{Z}'}{\mathbf{Z}}\frac{dE}{dx} - \mathbf{YZ}E = 0. \tag{21}$$

Before (21) can be solved we have to insert the values of $\mathbf{Y}$ and $\mathbf{Z}$ as functions of x, the distance from the zero point. Let us write $\mathbf{Z} = \mathbf{Z}_1 x^\alpha$ and $\mathbf{Y} = \mathbf{Y}_1 x^\beta$, where the indices α and β are constants. Using these values of $\mathbf{Y}$, $\mathbf{Z}$ in (21), we obtain [51]

$$\frac{d^2E}{dx^2} - \frac{\alpha}{x}\frac{dE}{dx} - k_1^2 x^\gamma E = 0, \tag{22}$$

where $\gamma = (\alpha+\beta)$ and $k_1^2 = \mathbf{Y}_1\mathbf{Z}_1$. Substituting $E = yz^p$ and $x = z^q$, we have

$$\frac{dE}{dx} = \frac{dE}{dz}\frac{dz}{dx} = \frac{z^{1-q}}{q}\{z^p y' + pz^{p-1}y\}; \tag{23}$$

so

$$-\frac{\alpha}{x}\frac{dE}{dx} = -\frac{z^{p-2q+2}}{q^2}\left\{\frac{\alpha}{z}qy' + \frac{\alpha pqy}{z^2}\right\}. \tag{24}$$

Also

$$\frac{d^2E}{dx^2} = \frac{d}{dz}\left(\frac{dE}{dx}\right)\frac{dz}{dx} = \frac{z^{1-q}}{q^2}\{z^{p-q+1}y''+(2p-q+1)z^{p-q}y'+p(p-q)z^{p-q-1}y\}$$

$$= \frac{z^{p-2q+2}}{q^2}\left\{y''+(2p-q+1)\frac{y'}{z}+p(p-q)\frac{y}{z^2}\right\}, \tag{25}$$

and
$$-k_1^2 x^\gamma E = -k_1^2 z^{q\gamma+p}y. \tag{26}$$

Adding (24), (25), (26) and dividing throughout by $\dfrac{z^{p-2q+2}}{q^2}$, we obtain

$$y''+[2p-q(\alpha+1)+1]\frac{y'}{z}-\left[q^2k_1^2 z^{q(2+\gamma)-2}+\frac{pq(\alpha+1)-p^2}{z^2}\right]y = 0. \tag{27}$$

Putting $2p-q(\alpha+1)+1 = 1$ and $q(2+\gamma)-2 = 0$, we get

$$p = (\alpha+1)/(\alpha+\beta+2) \quad \text{and} \quad q = 2/(\alpha+\beta+2).$$

Consequently (27) can be written in standard form, thus:

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\left(k^2+\frac{\nu^2}{z^2}\right)y = 0, \tag{28}$$

where $k = \dfrac{2k_1}{\alpha+\beta+2} = \dfrac{2\sqrt{(\mathbf{Y}_1\mathbf{Z}_1)}}{\alpha+\beta+2}$, and $\nu = p = \dfrac{\alpha+1}{\alpha+\beta+2}$.

From § 1 we see that the solution of (28) is

$$y = A_1 J_\nu(kzi)+B_1 J_{-\nu}(kzi), \tag{29}$$

when ν is fractional.

Since $E = yz^p$ and $x = z^q$, (29) can be written in the form

$$E = x^{\nu/q}\{A_1 J_\nu(kx^{1/q}i)+B_1 J_{-\nu}(kx^{1/q}i)\}. \tag{30}$$

To illustrate the preceding analysis, we shall consider some particular cases.

6. The uniform line

When $\alpha = \beta = 0$, $\mathbf{Z} = \mathbf{Z}_1$, $\mathbf{Y} = \mathbf{Y}_1$ both being constant throughout the cable. Also $\nu = \frac{1}{2}$ and $q = 1$, so

$$E = x^{\frac{1}{2}}\{A_1 J_{\frac{1}{2}}(kxi)+B_1 J_{-\frac{1}{2}}(kxi)\}. \tag{31}$$

Using formulae (28) and (28a), Chap. IV, we obtain

$$E = A_2 \sinh Px+B_2 \cosh Px, \tag{32}$$

where
$$P = k = \sqrt{(\mathbf{Y}_1\mathbf{Z}_1)} = \sqrt{\{(\mathbf{G}+i\omega\mathbf{C})(\mathbf{R}+i\omega\mathbf{L})\}},$$

this being a well-known result usually derived in a much simpler way.

7. The Heaviside Bessel line

In this case $\alpha = -1, \beta = 1$, so $\nu = p = 0, q = 1, \mathbf{Z} = \mathbf{Z}_1/x, \mathbf{Y} = \mathbf{Y}_1 x$ and $k = \sqrt{(\mathbf{Y}_1 \mathbf{Z}_1)}$. Thus from (30)

$$E = A_1 I_0(kx) + B_1 K_0(kx), \tag{33}$$

where k is complex and x is measured from 0. Suppose an e.m.f. E_0 is applied at $x = a$, and that the cable is earthed at $x = b$ where $a > b > 0$. Then

$$E_0 = A_1 I_0(ka) + B_1 K_0(ka)$$
$$0 = A_1 I_0(kb) + B_1 K_0(kb),$$

from which we obtain

$$A_1 = -E_0 K_0(kb)/[I_0(kb)K_0(ka) - I_0(ka)K_0(kb)],$$
$$B_1 = E_0 I_0(kb)/[I_0(kb)K_0(ka) - I_0(ka)K_0(kb)],$$

giving
$$E = E_0\left[\frac{-I_0(kx)K_0(kb) + K_0(kx)I_0(kb)}{I_0(kb)K_0(ka) - K_0(kb)I_0(ka)}\right]. \tag{34}$$

From (18)

$$I = -\frac{1}{\mathbf{Z}}\frac{dE}{dx} = -\frac{x}{\mathbf{Z}_1}\frac{dE}{dx} = E_0\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)}x\left[\frac{I_1(kx)K_0(kb) + K_1(kx)I_0(kb)}{I_0(kb)K_0(ka) - K_0(kb)I_0(ka)}\right]. \tag{35}$$

8. The linearly tapered line

In this case we have to introduce a complex origin since the impedance at the beginning of the tapered portion is not zero. At the head end of the line we can assume there is 160 nautical miles of unloaded cable, this being followed by one or more tapered loaded sections (Fig. 17), the system being electrically symmetrical about its geometrical centre. The series impedance per unit length at the beginning of a tapered section is $\mathbf{Z}_0 = \mathbf{R}_0 + i\omega\mathbf{L}_0$, whilst at a distance l therefrom, $\mathbf{Z} = \mathbf{Z}_0 + \mathbf{Z}_\Delta l$, where $\mathbf{Z}_\Delta = \mathbf{R}_\Delta + i\omega\mathbf{L}_\Delta$ the increase in impedance per unit length. If we write $\mathbf{Z} = \mathbf{Z}_\Delta x$, we get $x = \dfrac{\mathbf{Z}}{\mathbf{Z}_\Delta} = l + \dfrac{\mathbf{Z}_0}{\mathbf{Z}_\Delta}$. Since $\mathbf{Y}_0 = \mathbf{G}_0 + i\omega\mathbf{C}_0$ is substantially constant throughout the tapered section, we find that [52]

$$\alpha = 1 \quad \text{and} \quad \beta = 0, \quad \text{so} \quad \nu = p = \tfrac{2}{3}, \; q = \tfrac{2}{3}, \; k = \tfrac{2}{3}\sqrt{(\mathbf{Y}_0 \mathbf{Z}_\Delta)},$$

and from (30)
$$E = x\{A_1 J_{\frac{2}{3}}(kx^{\frac{3}{2}}i) + B_1 J_{-\frac{2}{3}}(kx^{\frac{3}{2}}i)\}. \tag{36}$$

From (18)

$$I = -\frac{1}{\mathbf{Z}}\frac{dE}{dx} = -\frac{1}{\mathbf{Z}_\Delta x}\frac{dE}{dx} = -\frac{\frac{3}{2}v}{\mathbf{Z}_\Delta x}\{A_1 J_{-\frac{1}{3}}(v) - B_1 J_{\frac{1}{3}}(v)\},$$

by differentiation and the use of recurrence formulae. ($v = kx^{\frac{3}{2}}i$.)

Thus $$I = \sqrt{\left(\frac{\mathbf{Y}_0}{\mathbf{Z}_\Delta}\right)} x^{\frac{1}{2}} i\{B_1 J_{\frac{1}{3}}(v) - A_1 J_{-\frac{1}{3}}(v)\}, \tag{37}$$
where $v = kx^{\frac{3}{2}} i$.

A complete solution of the problem where the cable has several sections (Fig. 17) involves the theory of electrical networks which is beyond the scope of this book [52]. By way of illustration, however, we shall find the current at the beginning of a very long isolated tapered cable. Here $l \to \infty$, so $x \to \infty$, the current $I \to 0$, and from (37)
$$B_1 = A_1 J_{-\frac{1}{3}}(v)/J_{\frac{1}{3}}(v). \tag{38}$$
To determine B_1 we use the asymptotic expansions, but it is first necessary to know in which quadrant v resides. Since
$$v = \tfrac{2}{3}\sqrt{(\mathbf{Y}_0 \mathbf{Z}_\Delta)}(l + \mathbf{Z}_0/\mathbf{Z}_\Delta)^{\frac{3}{2}} i$$
we see that when $l \to \infty$ the phase angle of $(l + \mathbf{Z}_0/\mathbf{Z}_\Delta)^{\frac{3}{2}}$ is zero. Thus the phase of v is $\theta = \frac{1}{2}\pi + \frac{1}{2}(\frac{1}{2}\pi - \tan^{-1}\mathbf{G}_0/\omega\mathbf{C}_0) + \frac{1}{2}(\frac{1}{2}\pi - \tan^{-1}\mathbf{R}_\Delta/\omega\mathbf{L}_\Delta)$. From data given below, $\mathbf{R}_\Delta \doteqdot 0$ whilst $\mathbf{G}_0 \ll \omega\mathbf{C}_0$, so $\theta = \pi - \epsilon$, where $\epsilon \ll 1$. Now when $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, we find from § 11, Chap. IV, that $J_\nu(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}\cos(z - \frac{1}{4}\pi - \frac{1}{2}\nu\pi)$. Following § 5, Chap. IV, and shifting the phase of v by $-\pi$ it becomes $-\epsilon$ and we write $z = ve^{-i\pi}$, which gives $J_\nu(ve^{-i\pi}) = e^{-i\nu\pi}J_\nu(v)$ or
$$J_\nu(v) = e^{i\nu\pi}J_\nu(z) \doteqdot \sqrt{\left(\frac{2}{\pi v}\right)} e^{i\nu\pi} e^{\frac{1}{2}\pi i}\cos(v + \tfrac{1}{4}\pi + \tfrac{1}{2}\nu\pi),$$
since $\cos x = \cos(-x)$. Applying this formula to (38) we obtain
$$B_1 = A_1\left[\frac{e^{-\frac{1}{3}\pi i}\cos(v + \frac{1}{12}\pi)}{e^{\frac{1}{3}\pi i}\cos(v + \frac{5}{12}\pi)}\right] = A_1 e^{-\frac{2}{3}\pi i}\left[\frac{e^{(v+\frac{1}{12}\pi)i} + e^{-(v+\frac{1}{12}\pi)i}}{e^{(v+\frac{5}{12}\pi)i} + e^{-(v+\frac{5}{12}\pi)i}}\right].$$
Since the imaginary part of v is positive and very large, the terms containing e^v are evanescent, so
$$B_1 = A_1 e^{-\frac{1}{3}\pi i} \quad \text{or} \quad A_1 = B_1 e^{\frac{1}{3}\pi i}. \tag{39}$$
When $l = 0$, $x_0 = \mathbf{Z}_0/\mathbf{Z}_\Delta$ and $E = E_0$, so from (36)
$$E_0 = \frac{\mathbf{Z}_0}{\mathbf{Z}_\Delta}\{e^{\frac{1}{3}\pi i}J_{\frac{2}{3}}(v_0) + J_{-\frac{2}{3}}(v_0)\}B_1,$$
or
$$B_1 = E_0 \Big/ \frac{\mathbf{Z}_0}{\mathbf{Z}_\Delta}[e^{\frac{1}{3}\pi i}J_{\frac{2}{3}}(v_0) + J_{-\frac{2}{3}}(v_0)]. \tag{40}$$
Substituting the values of A_1, B_1 from (39) and (40) in (37) and putting $x = x_0$, we obtain
$$I_0 = E_0\sqrt{\left(\frac{\mathbf{Y}_0}{\mathbf{Z}_0}\right)}\frac{1}{\varphi_0}, \tag{41}$$
where $$\varphi_0 = [e^{\frac{1}{3}\pi i}J_{\frac{2}{3}}(v_0) + J_{-\frac{2}{3}}(v_0)]/[J_{\frac{1}{3}}(v_0) - e^{\frac{1}{3}\pi i}J_{-\frac{1}{3}}(v_0)]i.$$

The quantity $Z_i = \varphi_0\sqrt{\left(\frac{Z_0}{Y_0}\right)}$ is known as the input impedance. For a uniform line $Z_i = \sqrt{\left(\frac{Z_0}{Y_0}\right)}$, so the influence of tapering the cable is to introduce the factor φ_0 whose value we shall now calculate, using the following data: $\omega/2\pi = 33 \sim$, $\mathbf{R}_0 = 4{\cdot}2$ ohms, $\mathbf{L}_0 = 87$ millihenrys, $\mathbf{G}_0 = 10^{-6}$ mho, $\mathbf{C}_0 = 0{\cdot}34$ microfarad, $\mathbf{R}_\Delta \doteqdot 0$, $\mathbf{L}_\Delta = 0{\cdot}586$ millihenry, all taken per nautical mile.

9. Calculation of $v_0 = \frac{2}{3}\frac{\mathbf{Y}_0^{\frac{1}{2}}\mathbf{Z}_0^{\frac{3}{2}}}{\mathbf{Z}_\Delta}i$

$$\mathbf{Y}_0 = \mathbf{G}_0+i\omega\mathbf{C}_0 \doteqdot 70{\cdot}5\times 10^{-6}i \quad \text{since } \mathbf{G}_0 \text{ is negligible.}$$

Thus
$$\mathbf{Y}_0^{\frac{1}{2}} = 8{\cdot}4\times 10^{-3}\,|\underline{45^\circ}. \tag{42}$$

$$\mathbf{Z}_0 = \mathbf{R}_0+i\omega\mathbf{L}_0 = 4{\cdot}2+18{\cdot}1i = 18{\cdot}6\,|\underline{76^\circ\,55'};$$

so
$$\mathbf{Z}_0^{\frac{3}{2}} = 80\,|\underline{115^\circ\,23'}. \tag{43}$$

$$\mathbf{Z}_\Delta = \mathbf{R}_\Delta+i\omega\mathbf{L}_\Delta = 0{\cdot}122\,|\underline{90^\circ}. \tag{44}$$

Hence
$$v_0 = \frac{2}{3}\frac{\mathbf{Y}_0^{\frac{1}{2}}\mathbf{Z}_0^{\frac{3}{2}}i}{\mathbf{Z}_\Delta} = \frac{2}{3}\times\frac{8{\cdot}4\times 10^{-3}\times 80}{0{\cdot}122}\,|\underline{45^\circ+115^\circ\,23'+90^\circ-90^\circ}$$
$$= 3{\cdot}67\,|\underline{160^\circ\,23'} \quad \text{(second quadrant).} \tag{45}$$

We also have
$$x_0 = \mathbf{Z}_0/\mathbf{Z}_\Delta = \frac{18{\cdot}6}{0{\cdot}122}\,|\underline{76^\circ\,55'-90^\circ}$$
$$= 152\,\overline{|13^\circ\,5'} \quad \text{(fourth quadrant),} \tag{46}$$

this giving the complex origin, the units being nautical miles.

10. Simplification of expression for φ_0 and numerical evaluation

Since v_0 is in the second quadrant we have from example 65, Chap. IV,

$$J_\nu(v_0) = e^{(\frac{1}{4}+\nu)\pi i}\sqrt{\left(\frac{2}{\pi v_0}\right)}[\zeta_\nu(v_0)\cos(u_0+\tfrac{1}{2}\nu\pi)-\xi_\nu(v_0)\sin(u_0+\tfrac{1}{2}\nu\pi)], \tag{47}$$

where $u_0 = (v_0+\frac{1}{4}\pi)$.

Applying this to the denominator of the expression for φ_0 and remembering that $\zeta_\nu = \zeta_{-\nu}$, $\xi_\nu = \xi_{-\nu}$, we obtain

$$\sqrt{\left(\frac{\pi v_0}{2}\right)}i[J_{\frac{1}{3}}(v_0)-e^{\frac{1}{3}\pi i}J_{-\frac{1}{3}}(v_0)] = e^{\frac{5}{6}\pi i}[\zeta_{\frac{1}{3}}\cos(u_0+\tfrac{1}{6}\pi)-\xi_{\frac{1}{3}}\sin(u_0+\tfrac{1}{6}\pi)]-$$
$$-e^{\pi i}[\zeta_{\frac{1}{3}}\cos(u_0-\tfrac{1}{6}\pi)-\xi_{\frac{1}{3}}\sin(u_0-\tfrac{1}{6}\pi)]$$
$$= \tfrac{1}{2}e^{u_0 i}[e^{\frac{2}{3}\pi i}-e^{\frac{5}{6}\pi i}][\zeta_{\frac{1}{3}}+i\xi_{\frac{1}{3}}], \tag{48}$$

this result being found by expressing the circular functions as exponentials.

In like manner we find that

$$\sqrt{\left(\frac{\pi v_0}{2}\right)}[e^{\frac{1}{3}\pi i}J_{\frac{2}{3}}(v_0)+J_{-\frac{2}{3}}(v_0)] = \tfrac{1}{2}e^{u_0 i}[e^{\frac{3}{2}\pi i}-e^{\frac{5}{6}\pi i}][\zeta_{\frac{2}{3}}+i\xi_{\frac{2}{3}}]. \tag{49}$$

Taking the ratio of (48) and (49) we get

$$\frac{1}{\varphi_0} = [\zeta_{\frac{1}{3}}(v_0)+i\xi_{\frac{1}{3}}(v_0)]/[\zeta_{\frac{2}{3}}(v_0)+i\xi_{\frac{2}{3}}(v_0)]. \tag{50}$$

Using the asymptotic expansions in § 11, Chap. IV, we have

$$\zeta_{\frac{1}{3}} \doteqdot 1-\frac{5\times 77}{162\times 64v_0^2};\ \xi_{\frac{1}{3}} \doteqdot -\frac{5}{72v_0};\ \zeta_{\frac{2}{3}} \doteqdot 1+\frac{7\times 65}{162\times 64v_0^2};\ \xi_{\frac{2}{3}} \doteqdot \frac{7}{72v_0}$$

from which

$$\frac{1}{\varphi_0} \doteqdot 1-\frac{0{\cdot}081}{v_0^2}-\frac{0{\cdot}167i}{v_0} = 1-\frac{0{\cdot}081}{13{\cdot}5}\overline{|320^\circ\,46'}-\frac{0{\cdot}167}{3{\cdot}67}\overline{|70^\circ\,23'}$$

using (45),

$$\doteqdot 1-0{\cdot}006\,\underline{|39^\circ\,14'}-0{\cdot}0455\,\overline{|70^\circ\,23'}$$

$$\doteqdot 0{\cdot}98+0{\cdot}039i = 0{\cdot}981\,\underline{|2^\circ\,17'}. \tag{51}$$

Hence substituting in (41) we obtain the current into the cable

$$I_0 = E_0\sqrt{\left(\frac{Y_0}{Z_0}\right)}0{\cdot}981\,\underline{|2^\circ\,17'}, \tag{52}$$

from which it appears that the influence of tapering is to reduce the current 1·9 per cent. and to advance its phase 2° 17′.

EXAMPLES.

1. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-k^2y = 0.$ [$y = A_1 I_0(kz)+B_1 K_0(kz)$.]

2. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-\left(3+\dfrac{1}{4z^2}\right)y = 0.$
[$y = A_1 I_{\frac{1}{2}}(z\sqrt{3})+B_1 K_{\frac{1}{2}}(z\sqrt{3})$, or $y = A_1 I_{\frac{1}{2}}(z\sqrt{3})+B_1 I_{-\frac{1}{2}}(z\sqrt{3})$.]

3. Solve $\dfrac{d^2u}{dr^2}+\dfrac{1}{r}\dfrac{du}{dr}-\left(ik^2+\dfrac{1}{r^2}\right)u = 0.$ [$u = A_1 I_1(kri^{\frac{1}{2}})+B_1 K_1(kri^{\frac{1}{2}})$.]

4. Solve $3z^3\dfrac{d^4y}{dz^4}+6z^2\dfrac{d^3y}{dz^3}-3z\dfrac{d^2y}{dz^2}+3\dfrac{dy}{dz}-3z^3k^4y = 0.$
[$y = A_1 J_0(kz)+B_1 Y_0(kz)+C_1 I_0(kz)+D_1 K_0(kz)$.]

5. Solve $\dfrac{\partial^2 v}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial v}{\partial r}-\dfrac{v}{r^2} = \dfrac{1}{f}\dfrac{\partial v}{\partial t}.$ [$u = \{A_1 I_1(kri^{\frac{1}{2}})+B_1 K_1(kri^{\frac{1}{2}})\}e^{i\omega t}$,
$k = \sqrt{(\omega/f)}$. Substitute $v = ue^{i\omega t}$, where u is a function of r, but not of t.]

6. The electric field E in the sea around a submarine cable is found from the differential equation [28]

$$\frac{\partial^2 E}{\partial r^2}+\frac{1}{r}\frac{\partial E}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E}{\partial \phi^2}-4\pi\lambda\mu\omega Ei = 0.$$

If $E = 0$ when r is infinite, i.e. the field is evanescent at an infinite distance from the cable, show that

$$E = A_0 K_0(kr)+A_1 K_1(kr)\cos\phi+A_2 K_2(kr)\cos 2\phi+\ldots,$$

where $k^2 = 4\pi\lambda\mu\omega i$. [Substitute $E = \chi\cos n\phi$, where χ is a function of r but not of ϕ, and put $n = 0, 1, 2$, etc.]

7. Solve [63] the following equation, which occurs in a problem on heat conduction: $\dfrac{d^2\theta}{dr^2}+\dfrac{1}{r}\dfrac{d\theta}{dr}-k^2\theta = -k^2$. [$\theta = 1+A_1 I_0(kr)+B_1 K_0(kr)$.]

If $\dfrac{d\theta}{dr}+b\theta = 0$ when $r = a$, and θ is finite when $r = 0$, show that

$$\theta = 1-[bI_0(kr)/\{kI_1(ka)+bI_0(ka)\}].$$

8. Solve $\dfrac{d^2\psi}{dz^2}-\dfrac{1}{z}\dfrac{d\psi}{dz}-k^2\psi = 0$, where $\psi = yz$. [$\psi = z\{A_1 I_1(kz)+B_1 K_1(kz)\}$.]

9. Show that $\dfrac{d^2\phi}{dx^2}-\left[a^2+\dfrac{n(n+1)}{x^2}\right]\phi = 0$ can be written in the form

$$\frac{d^2\varpi}{dx^2}+\frac{1}{x}\frac{d\varpi}{dx}-\left[a^2+\frac{(n+\frac{1}{2})^2}{x^2}\right]\varpi = 0, \quad \text{where } \varpi = \phi x^{-\frac{1}{2}}.$$

Hence solve the original equation.

[$\phi = x^{\frac{1}{2}}\{A_1 I_{n+\frac{1}{2}}(ax)+B_1 I_{-n-\frac{1}{2}}(ax)\}$.]

10. Show that $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-\left(1+\dfrac{n^2}{z^2}\right)y = 0$ can be written

$$\frac{d^2\zeta}{dz^2}+\left(\frac{1-4\nu^2}{4z^2}-1\right)\zeta = 0, \qquad \text{where } \zeta = yz^{\frac{1}{2}}.$$

11. Solve [74] the following differential equation for the lateral vibrations of a conical bar

$$[z^{-1}\vartheta(\vartheta+2)]^2 y = y, \quad \text{where } \vartheta = z\frac{d}{dz}.$$

$\left[y = \dfrac{1}{z}\{A_1 J_2(2z^{\frac{1}{2}})+B_1 Y_2(2z^{\frac{1}{2}})+C_1 I_2(2z^{\frac{1}{2}})+D_1 K_2(2z^{\frac{1}{2}})\}\right.$. Factorize and substitute $v = z^{\frac{1}{2}}$, then $y = u/v^2$.$\left.\right]$

The original equation is equivalent to $\dfrac{d^2}{dz^2}\left(z^4\dfrac{d^2y}{dz^2}\right) = z^2y$. Confirm this equivalence.

2. The transverse vibrational amplitude of the axis of a conical bar [74] clamped at its base, $\xi = \xi_0[J_2(z)+C_1 I_2(z)]/z^2$, where $z = 2\sqrt{\{h(l-x)\}}$, x being the distance from the base, l the axial length, and h a dynamical constant. If $\xi = 0$ when $x = 0$, find C_1.

$$\left[C_1 = -\frac{J_2(2\sqrt{(hl)})}{I_2(2\sqrt{(hl)})}, \quad \text{so } \xi = \xi_0\left[J_2(z)-\left\{\frac{J_2(2\sqrt{(hl)})}{I_2(2\sqrt{(hl)})}\right\}I_2(z)\right]\Big/z^2.\right]$$

13. Show that

$$\phi = A_n I_n(nt)\sin n\theta;\ B_n I_n(nt)\cos n\theta;\ C_n K_n(nt)\sin n\theta;\ D_n K_n(nt)\cos n\theta$$

are solutions of $\left(t\frac{\partial}{\partial t}\right)^2\phi+(1+t^2)\frac{\partial^2\phi}{\partial\theta^2} = 0$, and write the solution in compact form.

[Expand the first term using the operator $\vartheta = t\partial/\partial t$ twice as indicated, then substitute $\phi = \chi\sin n\theta$, etc., where χ is a function of t but not of θ.

$$\sum_{n=1}^{\infty} a_n I_n(nt)\sin(n\theta+\alpha_n)+b_n K_n(nt)\sin(n\theta+\beta_n), \text{ where } a_n = \sqrt{(A_n^2+B_n^2)};$$
$$b_n = \sqrt{(C_n^2+D_n^2)},\ \alpha_n = \tan^{-1}B_n/A_n,\ \beta_n = \tan^{-1}D_n/C_n.]$$

14. Verify that (*a*) $I_0(zi) = J_0(z)$; (*b*) $I_{\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\sinh z$;

$$(c)\ I_{-\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\cosh z;\ (d)\ \int_0^{\frac{1}{2}\pi} I_0(z\cos\theta)\cos\theta\, d\theta = \frac{\sinh z}{z}.$$

[Compare (*b*) and (*c*) with (23), (24), (28), (28 a) in Chap. IV; compare (*d*) with example 17 (*b*), Chap. I.]

15. Verify that (*a*) $zK_1(z) \to 1$ as $z \to 0$; (*b*) $K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = \sqrt{\left(\frac{\pi}{2z}\right)}e^{-z}$.

16. If $R(\nu) > -1$ and $k^2 \neq l^2$, show that by writing li for l in (1), Chap. VI

$$\int_0^z J_\nu(kz)K_\nu(lz)z\,dz = \frac{z}{k^2+l^2}\{l\,J_\nu(kz)K_\nu'(lz)-k\,K_\nu(lz)J_\nu'(kz)\}+\frac{(k/l)^\nu}{k^2+l^2}$$
$$= \frac{z}{k^2+l^2}\{k\,K_\nu(lz)J_{\nu+1}(kz)-l\,J_\nu(kz)K_{\nu+1}(lz)\}+\frac{(k/l)^\nu}{k^2+l^2}$$
$$= -\frac{z}{k^2+l^2}\{k\,K_\nu(lz)J_{\nu-1}(kz)+l\,J_\nu(kz)K_{\nu-1}(lz)\}+\frac{(k/l)^\nu}{k^2+l^2}.$$

[The constant term is obtained on insertion of the lower limit, the approximate values of the functions when $z \to 0$ being used.]

17. When $R(\nu) > -1$ show that by putting $I_\nu(lz) = i^{-\nu}J_\nu(lzi)$ in (9), Chap. VI

$$\int_0^z J_\nu(kz)I_\nu(lz)z\,dz = \frac{z}{k^2+l^2}\{l\,J_\nu(kz)I_\nu'(lz)-k\,I_\nu(lz)J_\nu'(kz)\}$$
$$= \frac{z}{k^2+l^2}\{k\,I_\nu(lz)J_{\nu+1}(kz)+l\,J_\nu(kz)I_{\nu+1}(lz)\}$$
$$= -\frac{z}{k^2+l^2}\{k\,I_\nu(lz)J_{\nu-1}(kz)-l\,J_\nu(kz)I_{\nu-1}(lz)\}.$$

18. Show that the Wronskian determinant (§ 5, Chap. I),

$$W\{I_\nu(z),\ I_{-\nu}(z)\} = I_\nu^2(z)\frac{d}{dz}\left\{\frac{I_{-\nu}(z)}{I_\nu(z)}\right\}.$$

Given that the determinant is also equal to $-2\sin\nu\pi/\pi z$, show that

$$\int^z \frac{dz}{zI_\nu^2(z)} = -\frac{\pi}{2\sin\nu\pi}\frac{I_{-\nu}(z)}{I_\nu(z)}.$$

19. Given that

$$W\{K_\nu(z),\, I_\nu(z)\} = 1/z, \text{ show that } \int^z \frac{dz}{zK_\nu^2(z)} = I_\nu(z)/K_\nu(z).$$

[See example 18.]

20. Verify that $I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z) = 1/z$. [See example 19.]

21. Using the Wronskian determinant in example 19 show that

$$\int^z \frac{dz}{zK_\nu(z)I_\nu(z)} = \log\frac{I_\nu(z)}{K_\nu(z)}.$$

22. Verify that $u = I_0\{(z^2-y^2)^{\frac{1}{2}}\}$ is a solution of $\dfrac{\partial^2 u}{\partial z^2}-\dfrac{\partial^2 u}{\partial y^2}-u = 0$.

23. Show that $\dfrac{\partial^2\xi}{\partial\omega^2}+\dfrac{\partial^2\xi}{\partial\eta^2}-\xi = 0$ is satisfied by $\xi = I_0\{(\omega^2+\eta^2)^{\frac{1}{2}}\}$.

24. Show that $\dfrac{\partial^2 x}{\partial y\partial z}-\frac{1}{4}x = 0$ is satisfied by $x = I_0[\sqrt{\{(y-\alpha)(z-\beta)\}}]$.

25. Verify that $i^{\nu+1}H_\nu^{(1)}(zi) = [I_{-\nu}(z)-I_\nu(z)]/\sin\nu\pi$.

26. Show that $K_0(z) = \frac{1}{2}\pi i[\pm I_0(z)+iY_0(\pm zi)]$.

27. Verify that $K_{\pm\nu}(z) = \frac{1}{2}\pi e^{\frac{1}{2}\pi(\nu+1)i}H_\nu^{(1)}(ze^{\frac{1}{2}\pi i}) = -\frac{1}{2}\pi i^{-\nu+1}H_\nu^{(2)}(ze^{-\frac{1}{2}\pi i})$.

[See § 5, Chap. IV.]

28. Show that (a) $K_\nu(z) = \frac{1}{2}\pi i^{-\nu+1}H_{-\nu}^{(1)}(zi)$; (b) $K_\nu(ze^{i\pi}) = e^{-i\nu\pi}K_\nu(z)-i\pi I_\nu(z)$.

29. Show that $\dfrac{d}{dz}\{z^{-n}I_n(z)\} = z^{-n}I_{n+1}(z)$, and thence that

$$z^{-n}I_n(z) = \int^z z^{-n}I_{n+1}(z)\,dz.$$

[See formula (24), Chap. II, also (88) in final list.]

30. Show that $\dfrac{d}{dz}\{z^nI_n(z)\} = z^nI_{n-1}(z)$, and thence that

$$z^nI_n(z) = \int^z z^nI_{n-1}(z)\,dz.$$

[See formula (25), Chap. II, also (89) in final list.]

31. Show that if ν is finite and z (real) $\to\infty$,

(a) $\sqrt{(2\pi z)}e^{-z}I_\nu(z)\to 1$;

(b) $\sqrt{\left(\dfrac{2z}{\pi}\right)}e^zK_\nu(z)\to 1$. [Use formulae (85) and (114) in the final list.]

32. Given that $K_0(z) = \displaystyle\int_0^\infty \frac{\cos t\,dt}{\sqrt{(t^2+z^2)}}$, and that $\displaystyle\int_0^\infty e^{-tx}J_0(zx)\,dx = 1/\sqrt{(t^2+z^2)}$,

show that $K_0(z) = \displaystyle\int_0^\infty \frac{xJ_0(zx)}{1+x^2}\,dx$.

33. Given that $\int_0^\infty \cos az\, K_0(bz)\, dz = \frac{1}{2}\pi/\sqrt{(a^2+b^2)}$, show by differentiating that

$$(a)\ \int_0^\infty z \sin az\, K_0(bz)\, dz = \tfrac{1}{2}\pi a/(a^2+b^2)^{\frac{3}{2}};$$

$$(b)\ \int_0^\infty z \sin\left(\frac{mz}{n}\right) K_0\left(\frac{pz}{n}\right) dz = \tfrac{1}{2}\pi mn^2/(m^2+p^2)^{\frac{3}{2}}.$$

34. In the Heaviside Bessel line § 7, an e.m.f. E_0 is applied at a distance a from the origin $x = 0$ where the line is open. Show that the p.d. and current at any point $x < a$ are $E = E_0 \dfrac{I_0(kx)}{I_0(ka)}$; $I = -E_0\sqrt{\left(\dfrac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)}\, x \dfrac{I_1(kx)}{I_0(ka)}$.

35. Show that formulae (34), (35) in the text § 7, reduce to those in example 34, when $b = 0$.

36. In § 8 show that the current at a distance l from the beginning of the taper is $I_l = \dfrac{E_0\sqrt{(\mathbf{Y}_0 \mathbf{Z}_\Delta)}}{\mathbf{Z}_0 \varphi} x^{\frac{1}{2}}$, where $\varphi = [e^{\frac{1}{3}\pi i} J_{\frac{2}{3}}(v_0) + J_{-\frac{2}{3}}(v_0)]/i[J_{\frac{1}{3}}(v) - e^{\frac{1}{3}\pi i} J_{-\frac{1}{3}}(v)]$.

37. In § 10 show that the influence of taper is to increase the input impedance Z_i from $\sqrt{\left(\dfrac{\mathbf{Z}_0}{\mathbf{Y}_0}\right)}$ to $\sqrt{\left(\dfrac{\mathbf{Z}_0}{\mathbf{Y}_0}\right)} 1{\cdot}019\ \underline{|2^\circ\ 17'}$. Calculate the latter value.

[524 $\underline{|8^\circ\ 50'}$ ohms.]

38. By solving (18) and (19) in § 5 for I, show that

$$I = x^{\nu/q}\{A_1 J_\nu(kx^{1/q} i) + B_1 J_{-\nu}(kx^{1/q} i)\},$$

where $\quad \nu = (\beta+1)/(\alpha+\beta+2), \quad q = 2/(\alpha+\beta+2).$

39. An inductance L is connected between an alternating current generator and a tapered loaded cable earthed at its far end. The series impedance is $\mathbf{Z} = i\omega \mathbf{L}_\Delta x$ and the shunt admittance $\mathbf{Y} = i\omega \mathbf{C}_1/x$, per unit length. Show that the current at any point in the cable is given by

$$I = \frac{E_0}{i\omega L}\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)}\{K_1(kx_1)I_0(kx) + I_1(kx_1)K_0(kx)\}\Big/\Big\{K_1(kx_1)\Big[\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)} I_0(kx_0) - \\ -\frac{x_0}{i\omega L} I_1(kx_0)\Big] + I_1(kx_1)\Big[\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)} K_0(kx_0) + \frac{x_0}{i\omega L} K_1(kx_0)\Big]\Big\},$$

where E_0 = alternator e.m.f., x_0 and x are, respectively, the distances of the beginning and any point in the cable from the complex origin, $x_1 - x_0$ is the length of the cable, $k = \sqrt{(\mathbf{Y}_1 \mathbf{Z}_1)}$, $\mathbf{Y}_1 = i\omega \mathbf{C}_1$, $\mathbf{Z}_1 = i\omega \mathbf{L}_\Delta$.

40. In example 39 when k is small enough at low frequencies for the arguments of all the functions to approach zero, show that

$$I = E_0/i\omega\{L + \tfrac{1}{2}(x_1^2 - x_0^2)\mathbf{L}_\Delta\},$$

and interpret this result.

[Divide above and below by $K_1(kx_1)$ and use example 15 (a). The system behaves as a pure inductance whose value is $L + \frac{1}{2}(x_1^2 - x_0^2)\mathbf{L}_\Delta$, the current being in the same phase everywhere, i.e. there is absence of wave motion.]

41. In example 39 show that when $L = 0$, the current is given by

$$I = E_0\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)}[K_1(kx_1)I_0(kx)+I_1(kx_1)K_0(kx)]/x_0[K_1(kx_0)I_1(kx_1)-I_1(kx_0)K_1(kx_1)].$$

42. In example 41 show that the current is zero when $\dfrac{K_1(kx_1)}{I_1(kx_1)} = -\dfrac{K_0(kx)}{I_0(kx)}$,

and that when $|kx|$ and $|kx_1|$ are large enough this corresponds to

$$\omega = (2n+1)\pi/2(x_1-x)\surd(\mathbf{L}_\Delta\,\mathbf{C}_1).$$

[Use formulae (84) and (114) in final list.]

43. In example 41 show that the free periods are given by $\dfrac{K_1(kx_1)}{I_1(kx_1)} = \dfrac{K_1(kx_0)}{I_1(kx_0)}$,

this corresponding to $\omega = n\pi/(x_1-x_0)\surd(\mathbf{L}_\Delta\,\mathbf{C}_1)$, provided $|kx_0|$ and $|kx_1|$ are large enough. [Use formulae (84) and (114) in final list].

44. In example 41 show that when $|kx_0|$, $|kx_1|$, $|kx|$ are large enough the current into the cable is $I_0 = -iE_0\sqrt{\left(\frac{\mathbf{C}_1}{\mathbf{L}_\Delta}\right)}\cot\{\omega\surd(\mathbf{L}_\Delta\,\mathbf{C}_1)(x_1-x_0)\}$, and the current at any point x is

$$I = -iE_0\sqrt{\left(\frac{\mathbf{C}_1}{\mathbf{L}_\Delta}\right)}\sqrt{\frac{x_0}{x}}\,\frac{\cos\{\omega\surd(\mathbf{L}_\Delta\,\mathbf{C}_1)(x_1-x)\}}{\sin\{\omega\surd(\mathbf{L}_\Delta\,\mathbf{C}_1)(x_1-x_0)\}}.$$

The negative value of i signifies that the applied e.m.f. E_0 leads the current into the cable by 90°.

45. In example 39 show that when the far end of the cable is open (current there is zero) that the current at any point is

$$I = \frac{E_0}{i\omega L}\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)}\{K_0(kx_1)I_0(kx)-I_0(kx_1)K_0(kx)\}\Big/\Big\{K_0(kx_1)\Big[\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)}I_0(kx_0)-$$
$$-\frac{x_0}{i\omega L}I_1(kx_0)\Big]-I_0(kx_1)\Big[\sqrt{\left(\frac{\mathbf{Y}_1}{\mathbf{Z}_1}\right)}K_0(kx_0)+\frac{x_0}{i\omega L}K_1(kx_0)\Big]\Big\}.$$

46. If in example 44 the cable is considered to behave as an effective inductance, L_e, then $E_0 = i\omega L_e I_0$. Show that $L_e = \dfrac{1}{\omega}\sqrt{\left(\frac{\mathbf{L}_\Delta}{\mathbf{C}_1}\right)}\tan\{\omega\surd(\mathbf{L}_\Delta\,\mathbf{C}_1)(x_1-x_0)\}$, and plot a curve showing the variation in effective inductance with frequency.

[The above examples can be applied to the purely *longitudinal* vibrations of a solid or a hollow cone (loud-speaker diaphragm, gun). $E_0 = f_0$ the driving force, $I =$ particle velocity, $\mathbf{L}_\Delta = m_\Delta$ the mass change per unit length, $\mathbf{C}_1 = 1/s_1$ where s_1 is the stiffness constant [83]. Examples 39 and 45 correspond to a coil of mass $m = L$, driving free and clamped edge cones, respectively. In example 40 the cone and coil move as a whole. In example 44 the coil has negligible mass, whilst in example 46 $L_e = m_e$ the effective mass of a cone to longitudinal vibrations. In acoustical work the flexural vibrations are the more important of the two.]

VIII

BER, BEI, KER, AND KEI FUNCTIONS

1. Functions of order zero

In certain problems which occur in electrical engineering it is necessary to solve the differential equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-ik^2y = 0. \tag{1}$$

Putting $k_1^2 = -ik^2$, we see from Chapter I that the solution of (1) is

$$y = A_1 J_0(k_1 z)+B_1 Y_0(k_1 z). \tag{2}$$

Since $k_1 = \pm\sqrt{(-i)}k = \pm i^{\frac{3}{2}}k$, an ambiguity is introduced. To avoid

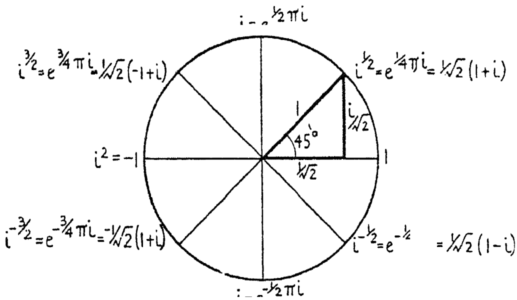


FIG. 18. Diagram illustrating $i^{\frac{1}{2}}$, $i^{\frac{3}{2}}$, $i^{-\frac{1}{2}}$, etc.

this we shall use $k_1 = +i^{\frac{3}{2}}k = ke^{\frac{3}{4}\pi i}$ in this book. Fig. 18 illustrates the complex quantities $i^{\frac{1}{2}}$, $i^{-\frac{1}{2}}$, $i^{\frac{3}{2}}$, $i^{-\frac{3}{2}}$, in accordance with the convention adopted herein. Thus the complete solution of (1) becomes

$$y = A_1 J_0(kzi^{\frac{3}{2}})+B_1 Y_0(kzi^{\frac{3}{2}}). \tag{3}$$

Now both $J_0(kzi^{\frac{3}{2}})$ and $Y_0(kzi^{\frac{3}{2}})$ are infinite when kz is infinite, which is an undesirable feature in the solution of practical problems. It is more suitable to take $K_0(kzi^{\frac{1}{2}})$ as the second solution, since it approaches zero asymptotically as $kz \to \infty$, and vice versa. Accordingly the solution of (1) takes the form

$$y = A_1 J_0(kzi^{\frac{3}{2}})+B_1 K_0(kzi^{\frac{1}{2}}), \tag{4}†$$

where the constants A_1 and B_1 differ from those in (3).

When $k = 1$, the solution of (1) is

$$y = A_1 J_0(zi^{\frac{3}{2}})+B_1 K_0(zi^{\frac{1}{2}}). \tag{5}$$

† From (6), Chap. VII, $K_0(kzi^{\frac{1}{2}}) = \frac{1}{2}\pi i H_0^{(1)}(kzi^{\frac{3}{2}})$.

From (22), Chap. I, we have

$$J_0(zi^{\frac{3}{2}}) = 1+i(\tfrac{1}{2}z)^2-\frac{(\frac{1}{2}z)^4}{(2!)^2}-\frac{i(\frac{1}{2}z)^6}{(3!)^2}+...$$

$$= \left\{1-\frac{(\frac{1}{2}z)^4}{(2!)^2}+\frac{(\frac{1}{2}z)^8}{(4!)^2}-...\right\}+i\left\{(\tfrac{1}{2}z)^2-\frac{(\frac{1}{2}z)^6}{(3!)^2}+...\right\}. \tag{6}$$

Following Kelvin [80] we call the series in the first bracket ber z, and that in the second bracket bei z (Bessel-real and Bessel-imaginary).

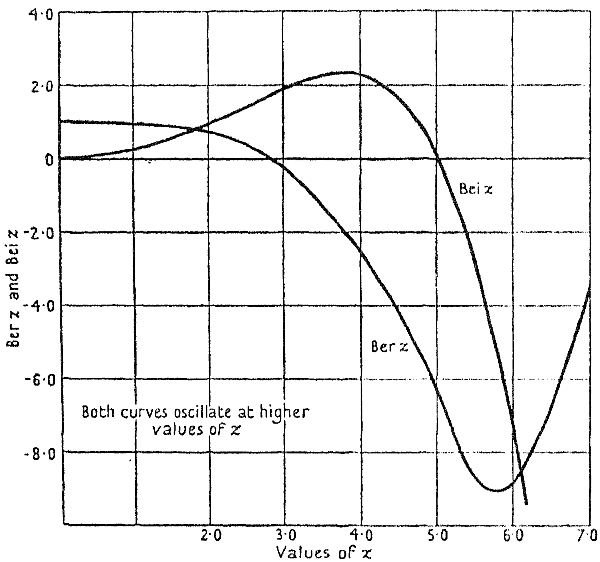


FIG. 19. The functions ber z and bei z, both of which oscillate about the z-axis.

Thus by definition we have

$$\text{ber}\, z = \left[1-\frac{(\frac{1}{2}z)^4}{(2!)^2}+\frac{(\frac{1}{2}z)^8}{(4!)^2}-...\right] = \left[1-\frac{z^4}{2^2.4^2}+\frac{z^8}{2^2.4^2.6^2.8^2}-...\right],$$

and (6 a)

$$\text{bei}\, z = \left[(\tfrac{1}{2}z)^2-\frac{(\frac{1}{2}z)^6}{(3!)^2}+...\right] = \left[\frac{z^2}{2^2}-\frac{z^6}{2^2.4^2.6^2}+...\right]. \tag{6 b}$$

These functions are plotted in Fig. 19. As z increases they both oscillate about the horizontal axis. It may be remarked that when z is real, $\lim_{z\to\infty} J_0(z) = 0$, but $\lim_{z\to\infty} |J_0(zi^{\frac{3}{2}})| = \infty$. From above we have

$$J_0(zi^{\frac{3}{2}}) = \text{ber}\, z+i\,\text{bei}\, z. \tag{7}$$

It can also be shown that

$$J_0(zi^{-\frac{3}{2}}) = \operatorname{ber} z - i \operatorname{bei} z. \tag{7 a}$$

From (7),
$$\frac{d}{dz}\{J_0(zi^{\frac{3}{2}})\} = \operatorname{ber}' z + i \operatorname{bei}' z,$$

and
$$J_0'(zi^{\frac{3}{2}}) = \frac{d}{d(zi^{\frac{3}{2}})} J_0(zi^{\frac{3}{2}}) = i^{-\frac{3}{2}}(\operatorname{ber}' z + i \operatorname{bei}' z). \tag{8}$$

The functions $\operatorname{ker} z$ and $\operatorname{kei} z$ derived in connexion with the second solution to (1), are defined as follows:

$$K_0(zi^{\frac{1}{2}}) = \operatorname{ker} z + i \operatorname{kei} z, \tag{9}$$

$$K_0(zi^{-\frac{1}{2}}) = \operatorname{ker} z - i \operatorname{kei} z, \tag{10}$$

where

$$\operatorname{ker} z = (\log 2 - \gamma - \log z)\operatorname{ber} z + \tfrac{1}{4}\pi \operatorname{bei} z - \\ - \frac{(\frac{1}{2}z)^4}{(2!)^2}(1+\tfrac{1}{2}) + \frac{(\frac{1}{2}z)^8}{(4!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}) - \dots \tag{11}$$

and

$$\operatorname{kei} z = (\log 2 - \gamma - \log z)\operatorname{bei} z - \tfrac{1}{4}\pi \operatorname{ber} z + \\ + (\tfrac{1}{2}z)^2 - \frac{(\frac{1}{2}z)^6}{(3!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \dots, \tag{12}$$

and $\log 2 - \gamma \doteqdot 0{\cdot}1159$, whilst Euler's constant $\gamma \doteqdot 0{\cdot}5772$. Asymptotic expansions of the above functions are given in the list of formulae, p. 172.

2. Functions of order ν

The functions $\operatorname{ber}_\nu z$, $\operatorname{bei}_\nu z$, $\operatorname{ker}_\nu z$, and $\operatorname{kei}_\nu z$ are defined as follows:

$$J_\nu(zi^{\frac{3}{2}}) = \operatorname{ber}_\nu z + i \operatorname{bei}_\nu z \tag{13}$$

$$J_\nu(zi^{-\frac{3}{2}}) = \operatorname{ber}_\nu z - i \operatorname{bei}_\nu z \tag{14}$$

$$i^{-\nu} K_\nu(zi^{\frac{1}{2}}) = \operatorname{ker}_\nu z + i \operatorname{kei}_\nu z \tag{15}$$

$$i^{\nu} K_\nu(zi^{-\frac{1}{2}}) = \operatorname{ker}_\nu z - i \operatorname{kei}_\nu z. \tag{16}$$

Throughout this chapter we shall assume that z is real.

As an illustration suppose we find the series for $\operatorname{ber}_1 z$ and $\operatorname{bei}_1 z$. From (13), $J_1(zi^{\frac{3}{2}}) = \operatorname{ber}_1 z + i \operatorname{bei}_1 z$, and from (2), Chap. II,

$$J_1(zi^{\frac{3}{2}}) = \frac{zi^{\frac{3}{2}}}{2}\left\{\left[1 - \frac{(\frac{1}{2}z)^4}{2!3!} + \dots\right] + i\left[\frac{(\frac{1}{2}z)^2}{1!2!} - \frac{(\frac{1}{2}z)^6}{3!4!} + \dots\right]\right\}. \tag{17}$$

Substituting for $i^{\frac{3}{2}}$ its value $\frac{1}{\sqrt{2}}(-1+i)$ from Fig. 18, we get on multi-

plication and separation into real and imaginary parts,

$$J_1(zi^{\frac{3}{2}}) = \frac{z}{2\sqrt{2}}\left\{\left[-1-\frac{(\frac{1}{2}z)^2}{1!2!}+\frac{(\frac{1}{2}z)^4}{2!3!}+\ldots\right]+i\left[1-\frac{(\frac{1}{2}z)^2}{1!2!}-\frac{(\frac{1}{2}z)^4}{2!3!}+\ldots\right]\right\}, \tag{18}$$

so
$$\operatorname{ber}_1 z = -\frac{z}{2\sqrt{2}}\left\{1+\frac{(\frac{1}{2}z)^2}{1!2!}-\frac{(\frac{1}{2}z)^4}{2!3!}-\frac{(\frac{1}{2}z)^6}{3!4!}+\ldots\right\}, \tag{19}$$

and
$$\operatorname{bei}_1 z = \frac{z}{2\sqrt{2}}\left\{1-\frac{(\frac{1}{2}z)^2}{1!2!}-\frac{(\frac{1}{2}z)^4}{2!3!}+\frac{(\frac{1}{2}z)^6}{3!4!}+\ldots\right\}. \tag{20}$$

These functions of order n are used in problems on proximity effects in stranded electrical conductors. Asymptotic expansions will be found in the final list of formulae, p. 169.

3. Representation of $J_\nu(zi^{\frac{3}{2}})$ and $J_\nu(zi^{-\frac{3}{2}})$ in the form $M_\nu(z)e^{\pm i\theta_\nu(z)}$

Since $J_\nu(zi^{\frac{3}{2}}) = \operatorname{ber}_\nu z + i\operatorname{bei}_\nu z$, it is frequently advantageous to express the function in terms of its modulus and phase. Thus we have

$$M_\nu(z)e^{i\theta_\nu(z)} = J_\nu(zi^{\frac{3}{2}}) = \sqrt{(\operatorname{ber}_\nu^2 z + \operatorname{bei}_\nu^2 z)}\{\cos\theta_\nu(z)+i\sin\theta_\nu(z)\}, \tag{21}$$

where
$$M_\nu(z) = \sqrt{(\operatorname{ber}_\nu^2 z + \operatorname{bei}_\nu^2 z)} \tag{22}$$
$$\operatorname{ber}_\nu z = M_\nu(z)\cos\theta_\nu(z) \tag{23}$$
$$\operatorname{bei}_\nu z = M_\nu(z)\sin\theta_\nu(z) \tag{24}$$
$$\theta_\nu(z) = \tan^{-1}\frac{\operatorname{bei}_\nu z}{\operatorname{ber}_\nu z}, \quad \text{as in Fig. 20 b.} \tag{25}$$

In particular
$$\operatorname{ber} z = M_0(z)\cos\theta_0(z) \tag{26}$$
$$\operatorname{bei} z = M_0(z)\sin\theta_0(z). \tag{27}$$

Similarly we write

$$i^{-\nu}K_\nu(zi^{\frac{1}{2}}) = \operatorname{ker}_\nu z+i\operatorname{kei}_\nu z \tag{28}$$
$$= \sqrt{(\operatorname{ker}_\nu^2 z + \operatorname{kei}_\nu^2 z)}\{\cos\phi_\nu(z)+i\sin\phi_\nu(z)\} \tag{29}$$
$$= N_\nu(z)e^{i\phi_\nu(z)}, \tag{30}$$

so
$$K_\nu(zi^{\frac{1}{2}}) = N_\nu(z)e^{i\{\phi_\nu(z)+\frac{1}{2}\nu\pi\}}, \tag{31}$$

where
$$N_\nu(z) = \sqrt{(\operatorname{ker}_\nu^2 z + \operatorname{kei}_\nu^2 z)} \tag{32}$$
$$\operatorname{ker}_\nu z = N_\nu(z)\cos\phi_\nu(z) \tag{33}$$
$$\operatorname{kei}_\nu z = N_\nu(z)\sin\phi_\nu(z) \tag{34}$$
$$\phi_\nu(z) = \tan^{-1}\frac{\operatorname{kei}_\nu z}{\operatorname{ker}_\nu z}. \tag{35}$$

In particular,
$$\operatorname{ker} z = N_0(z)\cos\phi_0(z) \tag{36}$$
$$\operatorname{kei} z = N_0(z)\sin\phi_0(z). \tag{37}$$

It can also be shown that

$$J_\nu(zi^{-\frac{3}{2}}) = M_\nu(z)e^{-i\theta_\nu(z)} = \mathrm{ber}_\nu\, z - i\,\mathrm{bei}_\nu\, z \tag{38}$$

and that
$$i^\nu K_\nu(zi^{-\frac{1}{2}}) = N_\nu(z)e^{-i\phi_\nu(z)} = \mathrm{ker}_\nu\, z - i\,\mathrm{kei}_\nu\, z, \tag{39}$$

so
$$K_\nu(zi^{-\frac{1}{2}}) = N_\nu(z)e^{-i\{\phi_\nu(z)+\frac{1}{2}\nu\pi\}}. \tag{40}$$

The above notation has the following advantages, although it cannot be used exclusively.

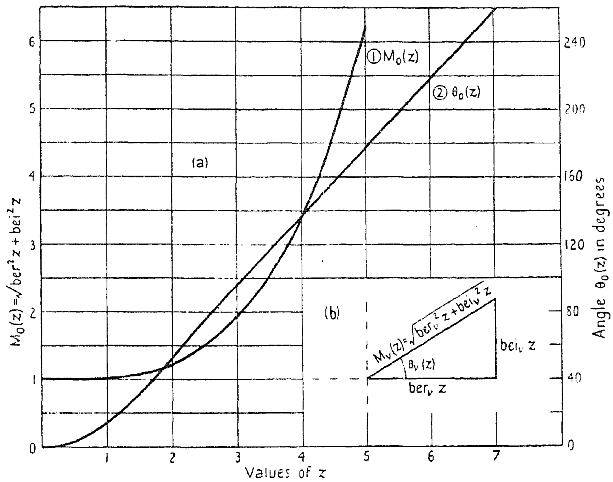


FIG. 20. (*a*) The functions $M_0(z) = \sqrt{(\mathrm{ber}^2 z+\mathrm{bei}^2 z)}$ and $\theta_0(z) = \tan^{-1}\dfrac{\mathrm{bei}\, z}{\mathrm{ber}\, z}$.
(*b*) Geometrical representation of $M_\nu(z)$ and $\theta_\nu(z)$.

(*a*) Formulae can be expressed briefly and concisely.

(*b*) The values of $M_\nu(z)$, $N_\nu(z)$, $\theta_\nu(z)$, $\phi_\nu(z)$ vary regularly as z increases, instead of oscillating as in the case of ber, bei, ker, and kei functions. (Compare Figs. 19 and 20 a.)

(*c*) It is equivalent to dealing with the magnitude (max. value or modulus) and phase of the physical quantity under consideration, instead of working with two components. For example, the curves of $M_0(z)$ and $\theta_0(z)$ in Fig. 20 show at once the variation in current density and phase in a straight circular wire, carrying alternating current, relative to the respective values at the axis.

(*d*) The tables can be arranged so that interpolation by proportional parts is possible throughout.

4. Relationship between $\mathrm{ber}'_\nu z$, $\mathrm{bei}'_\nu z$ and $M_\nu(z)e^{i\theta_\nu(z)}$; $\mathrm{ker}'_\nu z$, $\mathrm{kei}'_\nu z$ and $N_\nu(z)e^{i\phi_\nu(z)}$

From (20), Chap. II, $\qquad J'_0(z) = -J_1(z),$

so $$J'_0(zi^{\frac{3}{2}}) = -J_1(zi^{\frac{3}{2}}) = -M_1(z)e^{i\theta_1(z)},$$

and, therefore, $$\frac{1}{i^{\frac{3}{2}}}\frac{d}{dz}\{J_0(zi^{\frac{3}{2}})\} = -M_1(z)e^{i\theta_1(z)},$$

or $$\frac{d}{dz}J_0(zi^{\frac{3}{2}}) = \mathrm{ber}'z+i\,\mathrm{bei}'z = M_1(z)e^{i(\theta_1-\frac{1}{4}\pi)}, \tag{41}$$

since from Fig. 18, $i^{\frac{3}{2}} = -e^{-\frac{1}{4}\pi i}$.

Equating real and imaginary parts in (41), we obtain

$$\mathrm{ber}'z = M_1\cos(\theta_1-\tfrac{1}{4}\pi) \tag{42}$$

$$\mathrm{bei}'z = M_1\sin(\theta_1-\tfrac{1}{4}\pi). \tag{43}$$

It can also be shown by aid of formula (22), Chap. II, that

$$\mathrm{ber}'_\nu z = \tfrac{1}{2}\{M_{\nu+1}\cos(\theta_{\nu+1}-\tfrac{1}{4}\pi)-M_{\nu-1}\cos(\theta_{\nu-1}-\tfrac{1}{4}\pi)\}, \tag{44}$$

and $$\mathrm{bei}'_\nu z = \tfrac{1}{2}\{M_{\nu+1}\sin(\theta_{\nu+1}-\tfrac{1}{4}\pi)-M_{\nu-1}\sin(\theta_{\nu-1}-\tfrac{1}{4}\pi)\}. \tag{45}$$

By using (19) and (21), Chap. II, $\mathrm{ber}'_\nu z$ and $\mathrm{bei}'_\nu z$ can be expressed differently from (44) and (45). In addition we have

$$M_{-n}(z) = M_n(z) \tag{46}$$

$$\theta_{-n}(z) = \theta_n(z)+n\pi. \tag{47}$$

For the ker and kei functions it is easy to show that

$$\mathrm{ker}'z = N_1\cos(\phi_1-\tfrac{1}{4}\pi) \tag{48}$$

$$\mathrm{kei}'z = N_1\sin(\phi_1-\tfrac{1}{4}\pi) \tag{49}$$

$$\mathrm{ker}'_\nu z = \tfrac{1}{2}\{N_{\nu+1}(z)\cos(\phi_{\nu+1}-\tfrac{1}{4}\pi)-N_{\nu-1}(z)\cos(\phi_{\nu-1}-\tfrac{1}{4}\pi)\} \tag{50}$$

$$\mathrm{kei}'_\nu z = \tfrac{1}{2}\{N_{\nu+1}(z)\sin(\phi_{\nu+1}-\tfrac{1}{4}\pi)-N_{\nu-1}(z)\sin(\phi_{\nu-1}-\tfrac{1}{4}\pi)\} \tag{51}$$

$$N_{-n}(z) = N_n(z) \tag{52}$$

$$\phi_{-n}(z) = \phi_n(z)+n\pi. \tag{53}$$

5. Combinations of ber, bei, ker, and kei functions

In practical problems chiefly encountered by electrical engineers, these functions occur as sums and as products which can be represented in the $Me^{i\theta}$ notation. Thus

$$\mathrm{ber}^2z+\mathrm{bei}^2z = M_0^2(z) \tag{54}$$

$$\mathrm{ker}^2z+\mathrm{kei}^2z = N_0^2(z) \tag{55}$$

$$\mathrm{ber}'^2z+\mathrm{bei}'^2z = M_1^2(z) \tag{56}$$

$$\mathrm{ker}'^2z+\mathrm{kei}'^2z = N_1^2(z) \tag{57}$$

$$\operatorname{ber} z \operatorname{bei}' z - \operatorname{bei} z \operatorname{ber}' z = M_0(z) M_1(z) \sin(\theta_1 - \theta_0 - \tfrac{1}{4}\pi) \tag{58}$$

$$\operatorname{ber} z \operatorname{ber}' z + \operatorname{bei} z \operatorname{bei}' z = M_0(z) M_1(z) \cos(\theta_1 - \theta_0 - \tfrac{1}{4}\pi) \tag{59}$$

$$\operatorname{bei}' z \operatorname{ker}' z - \operatorname{ber}' z \operatorname{kei}' z = M_1(z) N_1(z) \sin(\theta_1 - \phi_1) \tag{60}$$

$$\operatorname{ber}' z \operatorname{ker}' z + \operatorname{bei}' z \operatorname{kei}' z = M_1(z) N_1(z) \cos(\theta_1 - \phi_1). \tag{61}$$

In solving problems by aid of ber, bei, ker, and kei functions, the notation to be used depends upon the type of problem. Sometimes it is convenient to work in ber, bei and then express the results in $Me^{i\theta}$ notation. Also there are cases where the $Me^{i\theta}$ notation can be used exclusively, and finally converted to ber, bei if desired. A series of examples will put the matter in more concrete form.

6. Example

Prove $\quad z \operatorname{bei}' z = z \frac{d}{dz} \operatorname{bei} z = z M_1 \sin(\theta_1 - \tfrac{1}{4}\pi) = \int^z z \operatorname{ber} z \, dz,$

or

$$z \operatorname{ber} z = \frac{d}{dz}(z \operatorname{bei}' z). \tag{62}$$

From (6 a)

$$\int^z z \operatorname{ber} z \, dz = \int^z \left[z - \frac{z^5}{2^2 . 4^2} + \frac{z^9}{2^2 . 4^2 . 6^2 . 8^2} \cdots \right] dz$$

$$= \frac{z^2}{2} - \frac{z^6}{2^2 . 4^2 . 6} + \frac{z^{10}}{2^2 . 4^2 . 6^2 . 8^2 . 10} - \dots . \tag{62a}$$

Differentiating (6 b) and multiplying by z, the series in (62 a) is reproduced, so the problem is solved. In like manner it can be proved that

$$\int^z z \operatorname{bei} z \, dz = -z \operatorname{ber}' z = -z M_1 \cos(\theta_1 - \tfrac{1}{4}\pi)$$

or

$$z \operatorname{bei} z = -\frac{d}{dz}(z \operatorname{ber}' z), \tag{63}$$

$$\int^z z \operatorname{ker} z \, dz = z \operatorname{kei}' z = z N_1(z) \sin(\phi_1 - \tfrac{1}{4}\pi)$$

or

$$z \operatorname{ker} z = \frac{d}{dz}(z \operatorname{kei}' z), \tag{64}$$

$$\int^z z \operatorname{kei} z \, dz = -z \operatorname{ker}' z = -z N_1(z) \cos(\phi_1 - \tfrac{1}{4}\pi)$$

or

$$z \operatorname{kei} z = -\frac{d}{dz}(z \operatorname{ker}' z). \tag{65}$$

These integrals represent the areas included between the z-axis and the curves $z\operatorname{ber} z$, $z\operatorname{bei} z$, and so on.

7. Example

Evaluate
$$\int_0^a z\operatorname{bei}(kz)\,dz.$$

Multiplying above and below by k^2, we get

$$\frac{1}{k^2}\int_0^a (kz)\operatorname{bei} kz\,d(kz) = -\frac{1}{k^2}\big[kz\operatorname{ber}'(kz)\big]_0^a$$
$$= -\frac{a^2\operatorname{ber}' ka}{ka}. \tag{66}$$

This corresponds to

$$\int_0^a z\,J_0(kz)\,dz = \frac{a^2 J_1(ka)}{ka} = -\frac{a^2 J_0'(ka)}{ka}. \tag{67}$$

8. Example

Express $\dfrac{d}{dz}\{J_0(zi^{-\frac{3}{2}})\}$ in $Me^{i\theta}$ notation.

$$\frac{1}{i^{-\frac{3}{2}}}\frac{d}{dz}J_0(zi^{-\frac{3}{2}}) = J_0'(zi^{-\frac{3}{2}}) = -J_1(zi^{-\frac{3}{2}}) = -M_1(z)e^{-i\theta_1(z)},$$

so
$$\frac{d}{dz}\{J_0(zi^{-\frac{3}{2}})\} = \operatorname{ber}'z - i\operatorname{bei}'z = M_1(z)e^{-i\{\theta_1(z)-\frac{1}{4}\pi\}}, \tag{68}$$

since from Fig. 18 $\quad i^{-\frac{3}{2}} = +e^{-\frac{3}{4}\pi i} = -e^{\frac{1}{4}\pi i}.$

By equating real and imaginary parts in (68) the results in (42) and (43) are reproduced.

9. Example

Evaluate
$$\int^z (\operatorname{ber}^2 z + \operatorname{bei}^2 z)z\,dz = \int^z zM_0^2(z)\,dz.$$

From (7) and (7 a) the integral can be written

$$\int^z z\,J_0(zi^{\frac{3}{2}})J_0(zi^{-\frac{3}{2}})\,dz.$$

From (8), Chap. VI, the integral is

$$-\frac{z}{2i}\left[J_0(zi^{\frac{3}{2}})\frac{d}{dz}\{J_0(zi^{-\frac{3}{2}})\} - J_0(zi^{-\frac{3}{2}})\frac{d}{dz}\{J_0(zi^{\frac{3}{2}})\}\right], \tag{69}$$

since $1/(i^3 - i^{-3}) = -1/2i$.

Substituting from (21), (38), (41), and (68) in (69) and reversing the order of the terms, we obtain

$$\frac{z}{2i}\{M_0\,e^{-i\theta_0}M_1\,e^{i(\theta_1-\frac{1}{4}\pi)}-M_0\,e^{i\theta_0}M_1\,e^{-i(\theta_1-\frac{1}{4}\pi)}\}$$
$$=\frac{zM_0\,M_1}{2i}\{e^{i(\theta_1-\theta_0-\frac{1}{4}\pi)}-e^{-i(\theta_1-\theta_0-\frac{1}{4}\pi)}\},$$

so
$$\int^{z} z(\mathrm{ber}^2 z+\mathrm{bei}^2 z)\,dz$$
$$= zM_0\,M_1\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi) = zM_0\,M_1\cos(\theta_0-\theta_1+\tfrac{3}{4}\pi). \quad (70)$$

Expanding the circular function, we get

$$\int^{z} = zM_0\,M_1\{\cos\theta_0\sin(\theta_1-\tfrac{1}{4}\pi)-\sin\theta_0\cos(\theta_1-\tfrac{1}{4}\pi)\} \quad (71)$$
$$= z\{\mathrm{ber}\,z\,\mathrm{bei}'z-\mathrm{bei}\,z\,\mathrm{ber}'z\} \quad (72)$$

from (26), (27), (42), and (43). As an alternative suppose we proceed from (69) onwards in terms of ber and bei functions: we get

$$\tfrac{1}{2}iz\{(\mathrm{ber}\,z+i\,\mathrm{bei}\,z)(\mathrm{ber}'z-i\,\mathrm{bei}'z)-(\mathrm{ber}\,z-i\,\mathrm{bei}\,z)(\mathrm{ber}'z+i\,\mathrm{bei}'z)\},$$

and after some multiplication the result in (72) is reproduced. It can also be shown that

$$\int_0^a (\mathrm{ber}^2 kr+\mathrm{bei}^2 kr)r\,dr = \frac{a}{k}\{\mathrm{ber}\,ka\,\mathrm{bei}'ka-\mathrm{bei}\,ka\,\mathrm{ber}'ka\} \quad (73)$$
$$= \frac{a}{k}M_0(ka)M_1(ka)\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi). \quad (73\,\mathrm{a})$$

10. Example [25]

If $\xi_n(z)-i\chi_n(z) = J_{n+1}(zi^{\frac{3}{2}})/J_{n-1}(zi^{\frac{3}{2}})$, find $\xi_1(z)$ and $\chi_1(z)$. This problem arises in treating the alternating current resistance of stranded conductors. From (23), Chap. II, when $n = 1$ we have

$$J_2(zi^{\frac{3}{2}}) = \frac{2}{zi^{\frac{3}{2}}}J_1(zi^{\frac{3}{2}})-J_0(zi^{\frac{3}{2}}),$$

which on division by $J_0(zi^{\frac{3}{2}})$ and substitution of $\dfrac{1}{i^{\frac{3}{2}}} = e^{-\frac{3}{4}\pi i}$ (Fig. 18) gives

$$\frac{J_2(zi^{\frac{3}{2}})}{J_0(zi^{\frac{3}{2}})} = \frac{2e^{-\frac{3}{4}\pi i}}{z}\frac{J_1(zi^{\frac{3}{2}})}{J_0(zi^{\frac{3}{2}})}-1$$
$$=\frac{2}{z}\frac{M_1\,e^{i(\theta_1-\frac{3}{4}\pi)}}{M_0\,e^{i\theta_0}}-1,$$

or
$$\xi_1(z)-i\chi_1(z) = \frac{2}{z}\frac{M_1\,e^{i(\theta_1-\theta_0-\frac{3}{4}\pi)}}{M_0}-1.$$

Equating real and imaginary parts

$$\xi_1(z) = \frac{2}{z}\frac{M_1}{M_0}\cos(\theta_1-\theta_0-\tfrac{3}{4}\pi)-1 = \frac{2}{z}\frac{M_1}{M_0}\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi)-1$$

$$= \frac{2M_0M_1}{zM_0^2}[\sin(\theta_1-\tfrac{1}{4}\pi)\cos\theta_0-\cos(\theta_1-\tfrac{1}{4}\pi)\sin\theta_0]-1$$

$$= \frac{2}{z}\left[\frac{\operatorname{ber} z\operatorname{bei}'z-\operatorname{bei} z\operatorname{ber}'z}{\operatorname{ber}^2z+\operatorname{bei}^2z}\right]-1. \tag{74}$$

$$\chi_1(z) = -\frac{2}{z}\frac{M_1}{M_0}\sin(\theta_1-\theta_0-\tfrac{3}{4}\pi) = \frac{2}{z}\frac{M_1}{M_0}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)$$

$$= \frac{2}{z}\frac{M_0M_1}{M_0^2}[\cos(\theta_1-\tfrac{1}{4}\pi)\cos\theta_0+\sin(\theta_1-\tfrac{1}{4}\pi)\sin\theta_0]$$

$$= \frac{2}{z}\left\{\frac{\operatorname{ber} z\operatorname{ber}'z+\operatorname{bei} z\operatorname{bei}'z}{\operatorname{ber}^2z+\operatorname{bei}^2z}\right\}. \tag{75}$$

EXAMPLES

1. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-4iy=0$. Express the solution in four different ways.

$[y = A_1 J_0(2zi^{\frac{3}{2}})+B_1 Y_0(2zi^{\frac{3}{2}})$; $y = A_1 J_0(2zi^{\frac{3}{2}})+B_1 K_0(2zi^{\frac{1}{2}})$; $y = A_1 M_0(2z)e^{i\theta_0(2z)}+B_1 N_0(2z)e^{i\phi_0(2z)}$; $y = A_1(\operatorname{ber} 2z+i\operatorname{bei} 2z)+$ $+B_1(\operatorname{ker} 2z+i\operatorname{kei} 2z)$.]

2. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+9iy=0$.

$[y = A_1 J_0(3zi^{\frac{1}{2}})+B_1 K_0(3zi^{-\frac{1}{2}})$; $y = A_1 M_0(3z)e^{-i\theta_0(3z)}+$ $+B_1 N_0(3z)e^{-i\phi_0(3z)} = A_1(\operatorname{ber} 3z-i\operatorname{bei} 3z)+B_1(\operatorname{ker} 3z-i\operatorname{kei} 3z)$.]

3. Write down two differential equations whose solutions contain $\operatorname{ber}_\nu kz$, $\operatorname{bei}_\nu kz$, $\operatorname{ker}_\nu kz$, and $\operatorname{kei}_\nu kz$.

$\left[\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-\left(ik^2+\dfrac{\nu^2}{z^2}\right)y=0;\ \dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+\left(ik^2-\dfrac{\nu^2}{z^2}\right)y=0.\right]$

4. Show that (a) $J_0(zi^{\frac{1}{2}}) = J_0(zi^{-\frac{3}{2}})$; (b) $J_0(zi^{-\frac{1}{2}}) = J_0(zi^{\frac{3}{2}})$;
(c) $i^{-\nu}I_\nu(ze^{-\frac{1}{4}\pi i}) = J_\nu(ze^{-\frac{3}{4}\pi i})$.

5. Solve $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-\dfrac{ik^2y}{z}=0$. $[y = A_1 J_0(2kz^{\frac{1}{2}}i^{\frac{3}{2}})+B_1 K_0(2kz^{\frac{1}{2}}i^{\frac{1}{2}})$ $= A_1 M_0(2kz^{\frac{1}{2}})e^{i\theta_0(2kz^{\frac{1}{2}})}+B_1 N_0(2kz^{\frac{1}{2}})e^{i\phi_0(2kz^{\frac{1}{2}})}$. Substitute $z=v^2$.]

6. Solve $\dfrac{d^2v}{dr^2}+\dfrac{1}{r}\dfrac{dv}{dr}-\left(ik^2+\dfrac{1}{r^2}\right)v=0$ in terms of $Me^{i\theta}$ and $Ne^{i\phi}$ functions.

$[v = A_1 M_1(kr)e^{i\theta_1(kr)}+B_1 N_1(kr)e^{i\{\phi_1(kr)+\frac{1}{2}\pi\}} = A_1(\operatorname{ber}_1 kr+i\operatorname{bei}_1 kr)+$ $+B_1(-\operatorname{kei}_1 kr+i\operatorname{ker}_1 kr)$.]

7. Differentiate the following with respect to z: (a) $M_0(z)$; (b) $\theta_0(z)$; (c) $M_1(z)$; (d) $\theta_1(z)$; (e) $N_0(z)$; (f) $\phi_0(z)$.

$\Big[$(a) $M_1(z)\cos(\theta_1-\theta_0-\frac{1}{4}\pi)$; (b) $\dfrac{M_1(z)}{M_0(z)}\sin(\theta_1-\theta_0-\frac{1}{4}\pi)$;

(c) $M_0(z)\sin(\theta_1-\theta_0-\frac{1}{4}\pi)-\dfrac{M_1}{z}$; (d) $\dfrac{M_0(z)}{M_1(z)}\cos(\theta_1-\theta_0-\frac{1}{4}\pi)$;

(e) $N_1(z)\cos(\phi_1-\phi_0-\frac{1}{4}\pi)$; (f) $\dfrac{N_1(z)}{N_0(z)}\sin(\phi_1-\phi_0-\frac{1}{4}\pi)$.]

8. Express the results (a) and (b) in example 7 in terms of ber and bei functions. $\left[(a)\ \dfrac{\operatorname{ber} z \operatorname{ber}' z+\operatorname{bei} z \operatorname{bei}' z}{\sqrt{(\operatorname{ber}^2 z+\operatorname{bei}^2 z)}};\ (b)\ \dfrac{\operatorname{ber} z \operatorname{bei}' z-\operatorname{bei} z \operatorname{ber}' z}{\operatorname{ber}^2 z+\operatorname{bei}^2 z}.\right]$

9. Show that $\dfrac{dM_\nu(z)}{dz}=\frac{1}{2}\{M_{\nu+1}(z)\cos(\theta_{\nu+1}-\theta_\nu-\frac{1}{4}\pi)-M_{\nu-1}(z)\cos(\theta_{\nu-1}-\theta_\nu-\frac{1}{4}\pi)\}$.

10. Show that

$$\frac{d\theta_\nu(z)}{dz}=\frac{1}{2M_\nu(z)}\{M_{\nu+1}(z)\sin(\theta_{\nu+1}-\theta_\nu-\tfrac{1}{4}\pi)-M_{\nu-1}(z)\sin(\theta_{\nu-1}-\theta_\nu-\tfrac{1}{4}\pi)\}.$$

11. Show that $\dfrac{dM_\nu(z)}{dz}=M_{\nu-1}(z)\cos(\theta_\nu-\theta_{\nu-1}-\frac{3}{4}\pi)-\dfrac{\nu}{z}M_\nu(z)$.

12. Show that $\dfrac{d\theta_\nu(z)}{dz}=\dfrac{1}{M_\nu(z)}\{M_{\nu-1}(z)\cos(\theta_\nu-\theta_{\nu-1}-\frac{1}{4}\pi)\}$.

13. If $\xi_2(z)-i\chi_2(z)=J_3(zi^{\frac{3}{2}})/J_1(zi^{\frac{3}{2}})$, verify that [25]

$$\xi_2(z)=\frac{4}{z}\frac{M_0(z)}{M_1(z)}\cos(\theta_0-\theta_1+\tfrac{1}{4}\pi)-1$$

$$\chi_2(z)=\frac{4}{z}\frac{M_0(z)}{M_1(z)}\sin(\theta_0-\theta_1+\tfrac{1}{4}\pi)-8/z^2.$$

This problem is associated with skin effect in stranded conductors.

14. Plot $\operatorname{ber}' z=M_1(z)\cos\{\theta_1(z)-\frac{1}{4}\pi\}$, using Table 15.
15. Plot $\operatorname{bei}' z=M_1(z)\sin\{\theta_1(z)-\frac{1}{4}\pi\}$, using Table 15.
16. Verify that $M_0(z)M_1(z)\sin(\theta_1-\theta_0-\frac{1}{4}\pi)=\operatorname{ber} z \operatorname{bei}' z-\operatorname{bei} z \operatorname{ber}' z$.
17. Verify that $M_0(z)M_1(z)\cos(\theta_1-\theta_0-\frac{1}{4}\pi)=\operatorname{ber} z \operatorname{ber}' z+\operatorname{bei} z \operatorname{bei}' z$.
18. Verify that $M_1(z)N_1(z)\sin(\theta_1-\phi_1)=\operatorname{bei}' z \operatorname{ker}' z-\operatorname{ber}' z \operatorname{kei}' z$.
19. Differentiate $M_0^2(z)=\operatorname{ber}^2 z+\operatorname{bei}^2 z$.

[$2M_0M_1\cos(\theta_1-\theta_0-\frac{1}{4}\pi)=2(\operatorname{ber} z \operatorname{ber}' z+\operatorname{bei} z \operatorname{bei}' z)$.]

20. Prove that

$$\int_0^a zJ_0(kzi^{\frac{3}{2}})\,dz=-\frac{a}{k}M_1(ka)e^{i\{\theta_1(ka)+\frac{1}{4}\pi\}}=\frac{a}{k}\{\operatorname{bei}' ka-i\operatorname{ber}' ka\}.$$

21. Evaluate $\int^z M_1^2(z)z\,dz=\int^z(\operatorname{ber}'^2 z+\operatorname{bei}'^2 z)z\,dz$.

[$zM_0(z)M_1(z)\cos(\theta_1-\theta_0-\frac{1}{4}\pi)=z(\operatorname{ber} z \operatorname{ber}' z+\operatorname{bei} z \operatorname{bei}' z)$.]

22. Evaluate $\int^z N_1^2(z)z\,dz=\int^z(\operatorname{ker}'^2 z+\operatorname{kei}'^2 z)z\,dz$.

[$zN_0(z)N_1(z)\cos(\phi_1-\phi_0-\frac{1}{4}\pi)=z(\operatorname{ker} z \operatorname{ker}' z+\operatorname{kei} z \operatorname{kei}' z)$.]

23. Evaluate $\int^z M_0^2(z)z\,dz=\int^z(\operatorname{ber}^2 z+\operatorname{bei}^2 z)z\,dz$, using formulae (62), (63) and integration by parts.

[$zM_0(z)M_1(z)\sin(\theta_1-\theta_0-\frac{1}{4}\pi)=z(\operatorname{ber} z \operatorname{bei}' z-\operatorname{bei} z \operatorname{ber}' z)$.]

24. Evaluate $\int^z N_0^2(z)z\,dz = \int^z (\mathrm{ker}^2 z+\mathrm{kei}^2 z)z\,dz$, (a) by the method of worked example § 9 in the text, (b) by aid of (64) and (65).

$$[zN_0(z)N_1(z)\sin(\phi_1-\phi_0-\tfrac{1}{4}\pi) = z(\mathrm{ker}\,z\,\mathrm{kei}'z - \mathrm{kei}\,z\,\mathrm{ker}'z).]$$

25. Evaluate $\int^z M_1^2(kz)z\,dz = \int^z (\mathrm{ber}'^2 kz+\mathrm{bei}'^2 kz)z\,dz$.

$$\left[\frac{z}{k}M_0(kz)M_1(kz)\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi) = \frac{z}{k}\{\mathrm{ber}\,kz\,\mathrm{ber}'kz + \mathrm{bei}\,kz\,\mathrm{bei}'kz\}.\right]$$

26. Evaluate $\int^z M_\nu^2(kz)z\,dz = \int^z (\mathrm{ber}_\nu^2 kz+\mathrm{bei}_\nu^2 kz)z\,dz$.

$$\left[\frac{zM_\nu(kz)}{2k}\{M_{\nu+1}(kz)\sin(\theta_{\nu+1}-\theta_\nu-\tfrac{1}{4}\pi)+M_{\nu-1}(kz)\sin(\theta_\nu-\theta_{\nu-1}+\tfrac{1}{4}\pi)\}\right.$$
$$\left. = \frac{z}{k}\{\mathrm{ber}_\nu\,kz\,\mathrm{bei}'_\nu\,kz - \mathrm{bei}_\nu\,kz\,\mathrm{ber}'_\nu\,kz\}.\right]$$

27. Prove that $M_1(z)N_1(z)\cos(\theta_1-\phi_1) = \mathrm{ber}'z\,\mathrm{ker}'z + \mathrm{bei}'z\,\mathrm{kei}'z$.

28. Show that (a) $\mathrm{ker}'z = N_1(z)\cos(\phi_1-\tfrac{1}{4}\pi)$; (b) $\mathrm{kei}'z = N_1(z)\sin(\phi_1-\tfrac{1}{4}\pi)$.

29. Establish the relationships

(a) $\mathrm{bei}'_\nu z = \tfrac{1}{2}\{M_{\nu+1}(z)\sin(\theta_{\nu+1}-\tfrac{1}{4}\pi)-M_{\nu-1}(z)\sin(\theta_{\nu-1}-\tfrac{1}{4}\pi)\}$;

(b) $\mathrm{ker}'_\nu z = \tfrac{1}{2}\{N_{\nu+1}(z)\cos(\phi_{\nu+1}-\tfrac{1}{4}\pi)-N_{\nu-1}(z)\cos(\phi_{\nu-1}-\tfrac{1}{4}\pi)\}$.

30. Show that (a) $\lim\limits_{k\to 0}\dfrac{M_1(kr)}{k} = \tfrac{1}{2}r$; (b) $\lim\limits_{z\to 0}\dfrac{1}{z}\dfrac{M_1(z)}{M_0(z)} = \tfrac{1}{2}$;

(c) $\lim\limits_{z\to 0}\dfrac{1}{z}\dfrac{M_1(z)}{M_0(z)}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi) = 0$; (d) $\lim\limits_{z\to 0}\dfrac{z}{2}\dfrac{M_0(z)}{M_1(z)}\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi) = 1$;

(e) $\lim\limits_{z\to 0}\dfrac{4}{z}\dfrac{M_0(z)}{M_1(z)}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi) = 1$; (f) $(\theta_0-\theta_1) = \tfrac{1}{8}z^2-\tfrac{3}{4}\pi$, $(z\to 0)$.

31. Using the asymptotic values of the ber and bei functions, p. 169, show that
$$\lim_{z\to\infty}\{\theta_1(z)-\theta_0(z)\} = \tfrac{1}{2}\pi;$$
show also that in general
$$\lim_{z\to\infty}\{\theta_\nu(z)-\theta_{\nu-1}(z)\} = \tfrac{1}{2}\pi.$$

32. Using the asymptotic values of the ker and kei functions, p. 172, show that $\lim\limits_{z\to\infty}\{\phi_\nu(z)-\phi_{\nu-1}(z)\} = -\tfrac{1}{2}\pi$. Also show that $\lim\limits_{z\to 0} z\,\mathrm{ker}'z = -1$.

33. If $Q = K_0(mai^{\frac{1}{2}})J_1(mbi^{\frac{3}{2}})+iJ_0(mai^{\frac{3}{2}})K_1(mbi^{\frac{1}{2}})$, show that the square of the modulus
$$|Q|^2 = M_1^2(mb)N_0^2(ma)+M_0^2(ma)N_1^2(mb)-$$
$$-2M_0(ma)M_1(mb)N_0(ma)N_1(mb)\cos\{(\phi_0-\theta_0)_{ma}+(\theta_1-\phi_1)_{mb}\}.$$

34. Show that $J_1'(zi^{-\frac{3}{2}}) = -\dfrac{M_1(z)}{z}e^{-i(\theta_1-\frac{3}{4}\pi)}+M_0(z)e^{-i\theta_0}$.

35. Show that $J_1'(zi^{\frac{3}{2}}) = -\dfrac{M_1(z)}{z}e^{i(\theta_1-\frac{3}{4}\pi)}+M_0(z)e^{i\theta_0}$.

36. Show that $K_1'(zi^{\pm\frac{1}{2}}) = -\frac{N_1(z)}{z}e^{\pm i(\phi_1+\frac{1}{4}\pi)}-N_0(z)e^{\pm i\phi_0}$.

37. Show that the indefinite integrals

$$\int J_1(zi^{\mp\frac{3}{2}})K_1(zi^{\pm\frac{1}{2}})z\,dz = \tfrac{1}{2}z[M_0(z)N_1(z)e^{\pm i(\phi_1-\theta_0+\frac{1}{4}\pi)}-M_1(z)N_0(z)e^{\pm i(\phi_0-\theta_1-\frac{1}{4}\pi)}].$$

[Use (132) in the final list of formulae.]

38. In example 33 show that when ma and mb are large enough for the asymptotic formulae to be used, $|Q|^2 \doteqdot \cosh\sqrt{2}m(a-b)+\cos\sqrt{2}m(a-b)$.

[Examples 33 to 38 occur in finding the eddy current loss in a furnace with a tubular charge. A solid cylindrical charge is treated in Chap. IX.]

39. Verify that $k\int^z z\,\mathrm{ber}\,kz\,dz = z\,\mathrm{bei}'kz$, and thence that

$$\frac{d}{dz}\left\{z\frac{d}{dz}(\mathrm{bei}\,kz)\right\} = k^2z\,\mathrm{ber}\,kz.$$

40. Verify that $-k\int^z z\,\mathrm{bei}\,kz\,dz = z\,\mathrm{ber}'kz$, and thence that

$$\frac{d}{dz}\left\{z\frac{d}{dz}(\mathrm{ber}\,kz)\right\} = -k^2z\,\mathrm{bei}\,kz.$$

41. Show that relationships of the form in examples 39 and 40 hold for ker and kei functions. Verify that $-(z/k)\mathrm{ker}'kz \to 1/k^2$ when $z \to 0$ (see p. 180).

42. Compute $N_0(z)$ and $\phi_0(z)$ from the values given in Table 10, and plot these functions. Take $z = 1, 2,\ldots$.

43. Compute $N_1(z)$ and $\phi_1(z)$ from the values given in Table 13, and plot these functions. Take $z = 1, 2,\ldots$.

44. Compute $M_4(z)$ and $\theta_4(z)$ from the values given in Table 12, and plot these functions. Take $z = 0, 1, 2,\ldots$.

45. Compute $\mathrm{ber}_3(2)$ and $\mathrm{bei}_3(2)$ without the aid of Tables. Check the results from Table 12.

46. Prove that $\mathrm{bei}'z = \frac{1}{\sqrt{2}}(\mathrm{bei}_1 z-\mathrm{ber}_1 z)$.

[Use $\mathrm{bei}'z = M_1\sin(\theta_1-\frac{1}{4}\pi)$ and expand.]

47. Show that $\mathrm{ber}'z = \frac{1}{\sqrt{2}}(\mathrm{ber}_1 z+\mathrm{bei}_1 z)$.

[Use $\mathrm{ber}'z = M_1(z)\cos(\theta_1-\frac{1}{4}\pi)$ and expand.]

48. Show that (a) $\mathrm{ber}'z\,\mathrm{bei}'z = \frac{1}{2}(\mathrm{bei}_1^2 z-\mathrm{ber}_1^2 z)$;

(b) $\mathrm{ber}'^2z+\mathrm{bei}'^2z = \mathrm{ber}_1^2 z+\mathrm{bei}_1^2 z$.

49. Verify that (a) $\mathrm{ber}_1'z - \mathrm{bei}_1'z = \frac{1}{\sqrt{2}}(\mathrm{ber}_2 z - \mathrm{ber}\,z)$;

(b) $\mathrm{ber}_1'z + \mathrm{bei}_1'z = \frac{1}{\sqrt{2}}(\mathrm{bei}_2 z - \mathrm{bei}\,z)$.

[Use $2J_1'(z) = J_0(z)-J_2(z)$.]

50. Show that $\int^z \mathrm{ber}_1 z\,dz = \frac{1}{\sqrt{2}}(\mathrm{ber}\,z-\mathrm{bei}\,z-1) = \frac{M_0(z)}{\sqrt{2}}[\cos\theta_0-\sin\theta_0]-\frac{1}{\sqrt{2}}$.

51. Verify that (*a*) $\int^z z^n \operatorname{ber}_{n-1} z\, dz = -\frac{z^n}{\sqrt{2}}(\operatorname{ber}_n z - \operatorname{bei}_n z)$;

(*b*) $\int^z z^n \operatorname{bei}_{n-1} z\, dz = -\frac{z^n}{\sqrt{2}}(\operatorname{ber}_n z + \operatorname{bei}_n z)$.

[Use (25), Chap. II.]

52. Show that (*a*) $\operatorname{ber}_n z = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{n+2r}\cos\frac{3}{4}(n+2r)\pi}{r!\,(n+r)!}$;

(*b*) $\operatorname{bei}_n z = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{n+2r}\sin\frac{3}{4}(n+2r)\pi}{r!\,(n+r)!}$.

[Use formula (13) in this chapter and (16 a) in Chapter IV.]

53. Show that

(*a*) $\operatorname{ber}'_n z = \sum_{r=0}^{\infty} \frac{(-1)^r\frac{1}{2}(n+2r)(\frac{1}{2}z)^{n+2r-1}\cos\frac{3}{4}(n+2r)\pi}{r!\,(n+r)!}$;

(*b*) $\operatorname{bei}'_n z = \sum_{r=0}^{\infty} \frac{(-1)^r\frac{1}{2}(n+2r)(\frac{1}{2}z)^{n+2r-1}\sin\frac{3}{4}(n+2r)\pi}{r!\,(n+r)!}$.

[Differentiate the series in example 52.]

54. Show that (*a*) $\cos\frac{3}{4}(n+2r)\pi = (-1)^n\cos\frac{1}{4}(n+2r)\pi$;

(*b*) $\sin\frac{3}{4}(n+2r)\pi = (-1)^{n+1}\sin\frac{1}{4}(n+2r)\pi$;

hence write the formulae in example 52 in another way [37].

[(*a*) $\operatorname{ber}_n z = \sum_{r=0}^{\infty} \frac{(-1)^{n+r}(\frac{1}{2}z)^{n+2r}\cos\frac{1}{4}(n+2r)\pi}{r!(n+r)!}$;

(*b*) $\operatorname{bei}_n z = \sum_{r=0}^{\infty} \frac{(-1)^{n+r+1}(\frac{1}{2}z)^{n+2r}\sin\frac{1}{4}(n+2r)\pi}{r!\,(n+r)!}$.]

55. For large values of z

$$J_n(zi^{\frac{3}{2}}) = \frac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\, e^{i(z/\sqrt{2}-\frac{1}{8}\pi+\frac{1}{2}n\pi)}[\lambda_n(z)+i\chi_n(z)];$$

hence show that

$$\operatorname{ber}'_n z = \frac{1}{2}\frac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\Big\{[\lambda_{n+1}(z)+\lambda_{n-1}(z)]\cos\Big(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}n\pi\Big)- -[\chi_{n+1}(z)+\chi_{n-1}(z)]\sin\Big(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}n\pi\Big)\Big\}.$$

[Use $2J'_n(w) = J_{n-1}(w)-J_{n+1}(w)$.]

56. If, for large values of z,

$$i^{-n}K_n(zi^{\frac{1}{2}}) = \sqrt{\Big(\frac{\pi}{2z}\Big)}e^{-z/\sqrt{2}}e^{-i(z/\sqrt{2}+\frac{1}{8}\pi+\frac{1}{2}n\pi)}[\lambda_n(-z)+i\chi_n(-z)],$$

show that

$$\operatorname{ker}'_n z = -\frac{1}{2}\sqrt{\Big(\frac{\pi}{2z}\Big)}e^{-z/\sqrt{2}}\Big\{[\lambda_{n+1}(-z)+\lambda_{n-1}(-z)]\cos\Big(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}n\pi\Big)+ +[\chi_{n+1}(-z)+\chi_{n-1}(-z)]\sin\Big(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}n\pi\Big)\Big\}.$$

[Use $2K'_n(w) = -[K_{n-1}(w)+K_{n+1}(w)]$.]

57. When z is large enough and $-\frac{1}{2}\pi <$ phase $z < \frac{1}{2}\pi$,

$$I_n(z) = \frac{e^z}{\sqrt{(2\pi z)}}\left\{1-\frac{(4n^2-1^2)}{1!\,8z}+\frac{(4n^2-1^2)(4n^2-3^2)}{2!(8z)^2}-\ldots\right\};$$

hence show that $J_n(zi^{\frac{3}{2}}) = i^n I_n(zi^{\frac{1}{2}})$ is given by the expression in example 55, where

$$\lambda_n(z) = 1-\frac{(4n^2-1^2)}{1!\,8z}\cos\tfrac{1}{4}\pi+$$
$$+\ldots+\frac{(-1)^r(4n^2-1^2)(4n^2-3^2)\ldots\{4n^2-(2r-1)^2\}\cos\frac{1}{4}r\pi}{r!(8z)^r}+\ldots,$$

$$\chi_n(z) = \frac{(4n^2-1^2)}{1!\,8z}\sin\tfrac{1}{4}\pi-$$
$$-\ldots+\frac{(-1)^{r+1}(4n^2-1^2)(4n^2-3^2)\ldots\{4n^2-(2r-1)^2\}\sin\frac{1}{4}r\pi}{r!(8z)^r}+\ldots.$$

[Put $zi^{\frac{1}{2}} = \dfrac{z}{\sqrt{2}}(i+1)$ in the exponent, use $(i^{\frac{1}{2}})^{-r} = e^{-\frac{1}{4}r\pi i}$ in the series, and write $i^n = e^{\frac{1}{2}n\pi i}$.]

58. Using the series for $\lambda_n(z)$ and $\chi_n(z)$ in example 57, show that [37]

(*a*) $\frac{1}{2}[\lambda_{n+1}(z)+\lambda_{n-1}(z)]$

$$= 1-\frac{(4n^2+1.3)}{1!\,8z}\cos\tfrac{1}{4}\pi-\frac{(4n^2-1)(4n^2-3^2)(4n^2+5.7)\cos\frac{3}{4}\pi}{3!(8z)^3}+\ldots$$
$$+\frac{(-1)^r(4n^2-1^2)(4n^2-3^2)\ldots\{4n^2-(2r-3)^2\}\{4n^2+(2r-1)(2r+1)\}\cos\frac{1}{4}r\pi}{r!(8z)^r}+\ldots;$$

(*b*) $$\tfrac{1}{2}[\chi_{n+1}(z)+\chi_{n-1}(z)] = \frac{(4n^2+1.3)}{1!\,8z}\sin\tfrac{1}{4}\pi-\frac{(4n^2-1)(4n^2+3.5)\sin\frac{1}{2}\pi}{2!(8z)^2}+\ldots$$
$$+\frac{(-1)^{r+1}(4n^2-1^2)(4n^2-3^2)\ldots\{4n^2-(2r-3)^2\}\{4n^2+(2r-1)(2r+1)\}\sin\frac{1}{4}r\pi}{r!(8z)^r}+\ldots.$$

In the next eight examples show that when $z > 8$:

59. $M_0(z) \doteq \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\left\{1+\dfrac{1}{8\sqrt{2}z}+\dfrac{1}{256z^2}-\dfrac{133}{2048\sqrt{2}z^3}\right\}.$

60. $\theta_0(z) \doteq \dfrac{z}{\sqrt{2}}-\frac{1}{8}\pi-\dfrac{1}{8\sqrt{2}z}-\dfrac{1}{16z^2}-\dfrac{25}{384\sqrt{2}z^3}$, radians.

61. $M_1(z) \doteq \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\left\{1-\dfrac{3}{8\sqrt{2}z}+\dfrac{9}{256z^2}+\dfrac{327}{2048\sqrt{2}z^3}\right\}.$

62. $\theta_1(z) \doteq \dfrac{z}{\sqrt{2}}+\frac{3}{8}\pi+\dfrac{3}{8\sqrt{2}z}+\dfrac{3}{16z^2}+\dfrac{21}{128\sqrt{2}z^3}$, radians.

63. $N_0(z) \doteq \sqrt{\left(\dfrac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{1-\dfrac{1}{8\sqrt{2}z}+\dfrac{1}{256z^2}+\dfrac{133}{2048\sqrt{2}z^3}\right\}.$

64. $\phi_0(z) \doteq -\dfrac{z}{\sqrt{2}}-\frac{1}{8}\pi+\dfrac{1}{8\sqrt{2}z}-\dfrac{1}{16z^2}+\dfrac{25}{384\sqrt{2}z^3}$, radians.

65. $N_1(z) \doteq \sqrt{\left(\dfrac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{1+\dfrac{3}{8\sqrt{2}z}+\dfrac{9}{256z^2}-\dfrac{327}{2048\sqrt{2}z^3}\right\}.$

66. $\phi_1(z) \doteq -\dfrac{z}{\sqrt{2}}-\frac{5}{8}\pi-\dfrac{3}{8\sqrt{2}z}+\dfrac{3}{16z^2}-\dfrac{21}{128\sqrt{2}z^3}$, radians.

[Use formulae 185, 186, on p. 169 for examples 59–62, and 230, 231 on p. 172 for examples 63–6. Observe that the terms in z^{-1}, z^{-2}, z^{-3} in 63–6 are obtained by writing $(-z)$ for z in those of 59–62.]

IX

APPLICATION OF BER AND BEI FUNCTIONS TO THE RESISTANCE OF CONDUCTORS TO ALTERNATING CURRENT

1. Current density in wire

When an alternating current passes through a cylindrical conductor, the varying magnetic field induces eddy currents therein. These currents react on the main current and affect it in such a way that its

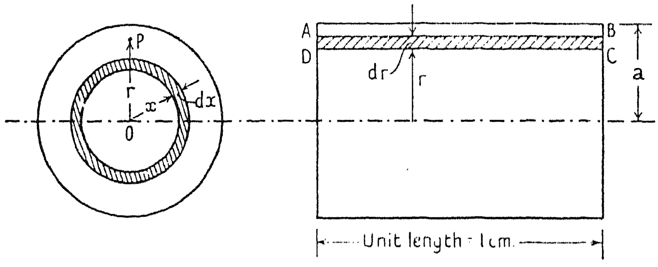


Fig. 21.

density increases from the centre of the wire towards the skin or outer surface. The current, so to speak, is driven to take the path of least impedance, which in this case is the outer portion of the wire. Hence the origin of the term 'skin-effect'. This electromagnetic effect, due to eddy currents, means that the available area is less than it would be with a steady unidirectional current. Hence there is an increase in effective resistance and a fall in effective inductance.

Fig. 21 a represents the cross-section of a very long isolated uniform cylindrical conductor, whose permeability and that of the surrounding medium is unity. Consider the magnetizing force at P due to the current in an elemental ring of radius x. The cross-sectional area of the ring is $2\pi x\,dx$, and the current passing through it parallel to the axis is $dI_x = 2\pi\sigma x\,dx$, where σ is the current density at radius x. The magnetizing force at P, due to this current ring is $dH = \frac{2dI_x}{r} = \frac{4\pi\sigma x\,dx}{r}$. Thus the total magnetic force at P due to the current within a radius r is $H = \frac{4\pi}{r}\int\limits^{r} \sigma x\,dx$. Differentiating

with respect to r, we obtain

$$\frac{dH}{dr} = 4\pi\sigma - \frac{4\pi}{r^2}\int_0^r \sigma x\,dx$$

$$= 4\pi\sigma - \frac{H}{r},$$

or

$$\frac{dH}{dr} + \frac{H}{r} - 4\pi\sigma = 0. \tag{1}$$

The p.d. per centimetre along the wire at a radius r is $E = \rho\sigma$, where ρ is the resistivity, whilst at $r+dr$ it is

$$E + \frac{dE}{dr}\,dr = \rho\left(\sigma + \frac{d\sigma}{dr}\,dr\right).$$

Thus the total e.m.f. round the circuit $ABCD$ is $-dE = -\rho\frac{d\sigma}{dr}dr$, and this must be equal to the rate of decrease of flux through the circuit. The flux through $ABCD$ (perpendicular to the paper) is $H\,dr$ per cm. length of wire and, therefore, $dE = \rho\frac{d\sigma}{dr}dr = \frac{dH}{dt}\,dr$. Thus

$$\rho\frac{d\sigma}{dr} = \frac{dH}{dt}. \tag{2}$$

We have now to construct a differential equation from (1) and (2) to determine σ at any radius r. Differentiating (1) with respect to t, we get

$$\frac{\partial^2 H}{\partial r\partial t} + \frac{1}{r}\frac{\partial H}{\partial t} - 4\pi\frac{\partial\sigma}{\partial t} = 0. \tag{3}$$

Using (2) this equation can be written

$$\frac{\partial^2\sigma}{\partial r^2} + \frac{1}{r}\frac{\partial\sigma}{\partial r} - \frac{4\pi}{\rho}\frac{\partial\sigma}{\partial t} = 0. \tag{4}$$

If the current is sinusoidal $\frac{\partial\sigma}{\partial t} = i\omega\sigma$. Substituting this value in (4) we obtain

$$\frac{d^2\sigma}{dr^2} + \frac{1}{r}\frac{d\sigma}{dr} - ik^2\sigma = 0, \tag{5}$$

where $k^2 = 4\pi\omega/\rho$. From (4), Chap. VIII, the solution of (5) is

$$\sigma = A_1 J_0(kri^{\frac{3}{2}}) + B_1 K_0(kri^{\frac{1}{2}}). \tag{6}$$

At the axis of the wire $r = 0$ but $K_0(kri^{\frac{1}{2}})$ is infinite, and since σ is finite $B_1 = 0$. The appropriate solution for our requirements is, therefore,

$$\sigma = A_1 J_0(kri^{\frac{3}{2}}) = A_1 M_0(kr)e^{i\theta_0(kr)}. \tag{7}$$

Putting $r = a$ in (7) we obtain $A_1 = \dfrac{\sigma_0 e^{-i\theta_0(ka)}}{M_0(ka)}$, where σ_0 is the current density at the skin of the wire. Substituting this value of A_1 in (7) we find the current density at any radius r to be

$$\sigma = \sigma_0 \frac{M_0(kr)}{M_0(ka)} e^{i\{\theta_0(kr)-\theta_0(ka)\}}. \tag{8}$$

In this expression the angle $\{\theta_0(kr)-\theta_0(ka)\}$ is the phase of σ at radius r relative to that at the skin. If the phase angle at the axis is taken

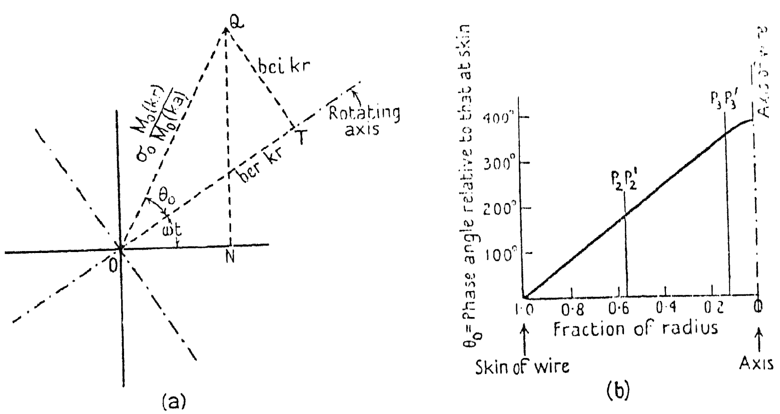


FIG. 22. (*a*) Geometrical representation of formula (9). (*b*) Showing phase angle of current, in a long straight copper wire 0·45 mm. radius, relative to that at skin, at $10^6\sim$. This corresponds to $\theta_0(z)$ of Fig. 20 if the origin is taken on axis.

to be zero we obtain

$$\sigma = \sigma_0 \frac{M_0(kr)}{M_0(ka)} e^{i\theta_0(kr)}. \tag{9}$$

Since $M_0(ka)$ and σ_0 are constant, the variation in σ with kr is seen at once from Fig. 20, curve 1, whilst the phase angle $\theta_0(kr)$ is shown in curve 2. From Fig. 20 or from Tables 14 and 15, it is clear that at a definite frequency the current density at a radius r lags behind that at the surface by a phase angle which increases continuously with decrease in r. If ka is large the current density vector may make several revolutions.

Expression (9) can be illustrated graphically, as shown in Fig. 22A, where OQ is $\sigma_0 \dfrac{M_0(kr)}{M_0(ka)}$, this being the maximum value of the current density at radius r. ON, the projection of OQ on the horizontal axis,

represents the current density σ at radius r at the epoch under consideration. The time axis OT rotates with angular velocity ω.

The phase angle θ_0 for a copper wire 0·45 mm. radius at $10^6 \sim$ is shown in Fig. 22 B. Over the radius it alters slightly in excess of 2π, so the current flows in opposite directions at different parts of the

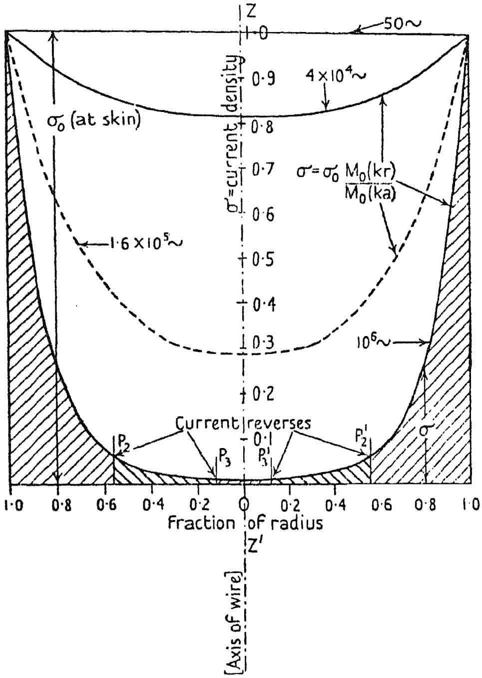


FIG. 23. Illustrating 'penetration' or current density at different radii in a long straight copper wire 0·45 mm. radius at various frequencies. The curves are all of the form $M_0(z) = \sqrt{(\mathrm{ber}^2 z + \mathrm{bei}^2 z)}$ on either side of the axis ZZ'. The points P_2, P_2', P_3, P_3' correspond to those in Fig. 22 B, each to each.

cross-section (see Fig. 23, curve for $10^6 \sim$). The net value of the total current is, therefore, less than it would be with zero phase angle throughout, so the alternating current resistance is increased correspondingly. The current density in the wire at several frequencies is shown in Fig. 23. At 50 $\sim$ it is uniformly distributed, but at $10^6 \sim$ the current is confined mainly to a skin layer whose thickness is some 20 per cent. of the radius.

2. Total current in wire

From (7), $\sigma_0 = A_1 J_0(kai^{\frac{3}{2}})$ at the surface where $r = a$, so

$$A_1 = \sigma_0 / J_0(kai^{\frac{3}{2}})$$

and, therefore,

$$\sigma = \sigma_0 \frac{J_0(kri^{\frac{3}{2}})}{J_0(kai^{\frac{3}{2}})}. \tag{10}$$

The current flowing along the wire through an elemental ring, 1 cm. long, of area $2\pi r\,dr$, is $2\pi\sigma r\,dr$, so the total current within a radius r is $I_r = 2\pi \int\limits_0^r \sigma r\,dr$. Substituting the value of σ from (10) in the integrand we have

$$I_r = \frac{2\pi\sigma_0}{J_0(kai^{\frac{3}{2}})} \int\limits_0^r J_0(kri^{\frac{3}{2}}) r\,dr$$

$$= \frac{2\pi\sigma_0 r}{ki^{\frac{3}{2}}} \frac{J_1(kri^{\frac{3}{2}})}{J_0(kai^{\frac{3}{2}})}$$

or

$$I_r = \frac{2\pi\sigma_0 r}{k} \frac{J_1(kri^{\frac{3}{2}})}{J_0(kai^{\frac{3}{2}})} e^{-\frac{3}{4}\pi i}. \tag{11}$$

Putting $r = a$, the total current in the wire is found to be

$$I = \frac{2\pi\sigma_0 a}{k} \frac{J_1(kai^{\frac{3}{2}})}{J_0(kai^{\frac{3}{2}})} e^{-\frac{3}{4}\pi i}. \tag{12}$$

In practice we are concerned with the total current I, and it will be convenient to express several quantities in terms of I or of the mean current density $\sigma_m = I/\pi a^2$, its relative phase being regarded as zero. From (12) above and (21), Chap. VIII,

$$\sigma_0 = \sigma_m \frac{ka}{2} \frac{M_0(ka)}{M_1(ka)} e^{i\{\theta_0(ka)-\theta_1(ka)+\frac{3}{4}\pi\}}. \tag{13}$$

Substituting this value of σ_0 in (10) and in (11) we obtain

$$\sigma = \sigma_m \frac{ka}{2} \frac{M_0(kr)}{M_1(ka)} e^{i\{\theta_0(kr)-\theta_1(ka)+\frac{3}{4}\pi\}} \tag{14}$$

and

$$I_r = I \frac{r}{a} \frac{M_1(kr)}{M_1(ka)} e^{-i\{\theta_1(ka)-\theta_1(kr)\}}. \tag{15}$$

It is seen from (13) that the current at the skin leads on the total current by a phase angle $\{\theta_0(ka)-\theta_1(ka)+\frac{3}{4}\pi\}$. Inspection of Tables 14 and 15 reveals that $\{\theta_0(ka)-\theta_1(ka)\}$ increases steadily from $-\frac{3}{4}\pi$ when $ka \doteqdot 0$ (low frequencies) towards $-\frac{1}{2}\pi$ when ka is very large (high frequencies). Under the latter condition the current at the skin leads the total current by $\frac{1}{4}\pi$.

The formulae given above can also be expressed in terms of ber and bei functions, as follows:

From (12)

$$I = \frac{2\pi\sigma_0 a}{ki^{\frac{1}{2}}} \frac{J_1(kai^{\frac{1}{2}})}{J_0(kai^{\frac{1}{2}})}$$

$$= -\frac{2\pi\sigma_0 a}{ki^{\frac{1}{2}}} \frac{J_0'(kai^{\frac{1}{2}})}{J_0(kai^{\frac{1}{2}})}$$

$$= \frac{2\pi\sigma_0 a}{k} \left[\frac{\mathrm{bei}'ka - i\,\mathrm{ber}'ka}{\mathrm{ber}\,ka + i\,\mathrm{bei}\,ka}\right] \text{ from § 1, Chap. VIII.} \tag{16}$$

From (16), $$\sigma_0 = \sigma_m \frac{ka}{2} \left[\frac{\mathrm{ber}\,ka + i\,\mathrm{bei}\,ka}{\mathrm{bei}'ka - i\,\mathrm{ber}'ka}\right], \tag{17}$$

where $\sigma_m = I/\pi a^2$.

From (10) and (17)

$$\sigma = \sigma_m \frac{ka}{2} \left[\frac{\mathrm{ber}\,kr + i\,\mathrm{bei}\,kr}{\mathrm{bei}'ka - i\,\mathrm{ber}'ka}\right]. \tag{18}$$

Substituting $J_1(kri^{\frac{1}{2}}) = -J_0'(kri^{\frac{1}{2}}) = i^{\frac{1}{2}}(\mathrm{ber}'kr + i\,\mathrm{bei}'kr)$, and the value of σ_0 from (16) in (11), we find that

$$I_r = I\frac{r}{a} \left[\frac{\mathrm{bei}'kr - i\,\mathrm{ber}'kr}{\mathrm{bei}'ka - i\,\mathrm{ber}'ka}\right]. \tag{19}$$

The mean square value of the total current over a complete cycle is from (16)

$$I^2_{\mathrm{r.m.s.}} = \tfrac{1}{2}I^2 = \frac{2\pi^2\sigma_0^2 a^2}{k^2} \left[\frac{\mathrm{ber}'^2 ka + \mathrm{bei}'^2 ka}{\mathrm{ber}^2 ka + \mathrm{bei}^2 ka}\right] \tag{20}$$

$$= \frac{2\pi^2\sigma_0^2 a^2}{k^2} \frac{M_1^2(ka)}{M_0^2(ka)}. \tag{21}$$

3. Example

If $ka = 10$ in the problem treated in § 2 above, show that the current in an outer ring of thickness $t = 4{\cdot}5/k$ is approximately 2·8 per cent. greater than the total current.

The inner radius of the ring is $r = (10-4{\cdot}5)/k = 5{\cdot}5/k$, so $kr = 5{\cdot}5$. Using formula (15) with Tables 14 and 15 we have

$$I_r = I\frac{5{\cdot}5}{10}\frac{7{\cdot}446}{144{\cdot}7}e^{-i(474{\cdot}28-293{\cdot}48)}$$

$$= 0{\cdot}0283Ie^{-180{\cdot}8i}$$

$$\doteqdot -0{\cdot}0283I.$$

The current within a radius r is, therefore, in opposite phase to the total current I, so the current in the outer ring must be $1{\cdot}0283I$.

4. Alternating current resistance of straight wire

This can be determined in two ways the simpler of which will be given first.

(a) Since the current does not induce an e.m.f. at the surface, the p.d. per cm. must be $\rho\sigma_0$. If R_e is the effective resistance and ωL_e the effective reactance of the wire per cm. length, we have

$$\rho\sigma_0 = I(R_e + i\omega L_e).$$

Using the value of σ_0 from (13) and inserting $\sigma_m = I/\pi a^2$, we obtain

$$R_e + i\omega L_e = \frac{\rho}{\pi a^2}\frac{ka}{2}\frac{M_0(ka)}{M_1(ka)}e^{i(\theta_0-\theta_1+\frac{3}{4}\pi)}.$$

Equating real and imaginary parts

$$R_e = \frac{\rho}{\pi a^2}\frac{ka}{2}\frac{M_0(ka)}{M_1(ka)}\cos(\theta_0-\theta_1+\tfrac{3}{4}\pi) \tag{22}$$

and
$$L_e = \frac{\rho}{\pi a^2}\frac{ka}{2\omega}\frac{M_0(ka)}{M_1(ka)}\sin(\theta_0-\theta_1+\tfrac{3}{4}\pi). \tag{23}$$

The latter does not represent the total inductance of the wire, but merely that per cm. length due to the flux within it for the chosen value of ka. Since the direct current resistance per cm. length is $\rho/\pi a^2$, the ratio

$$\frac{R_e}{R_{\text{d.c.}}} = \frac{ka}{2}\frac{M_0(ka)}{M_1(ka)}\cos(\theta_0-\theta_1+\tfrac{3}{4}\pi), \tag{24}$$

which can easily be computed by aid of Tables 14 and 15. When $ka > 4$, the ratio in (24) is approximately $\frac{1}{4}+\frac{ka}{2\sqrt{2}} = 0{\cdot}25+0{\cdot}354ka$.

(b) The second but more cumbersome method of determining R_e is to find the power dissipated as heat by an alternating current of prescribed r.m.s. value, and divide it by the expression for $I^2_{\text{r.m.s.}}$ in (20) or in (21).

The resistance of an elemental ring of radius r, and length 1 cm. is $\rho/2\pi r\,dr$. From (10) the current density in the ring is

$$\sigma = \sigma_0\frac{(\text{ber}\,kr+i\,\text{bei}\,kr)}{(\text{ber}\,ka+i\,\text{bei}\,ka)}.$$

The heat dissipated in the ring is, therefore,

$$d\mathbf{P}_1 = \rho[2\pi r\sigma_{\text{r.m.s.}}\,dr]^2/2\pi r\,dr.$$

Substituting for $\sigma_{\text{r.m.s.}}$ from above we get

$$d\mathbf{P}_1 = \frac{\pi\rho\sigma_0^2}{M_0^2(ka)}[\text{ber}^2kr+\text{bei}^2kr]r\,dr. \tag{25}$$

The heat dissipated in 1 cm. length of the wire is

$$\mathbf{P}_1 = \frac{\pi\sigma_0^2\rho}{M_0^2(ka)}\int_0^a (\mathrm{ber}^2 kr + \mathrm{bei}^2 kr) r\,dr$$

$$= \frac{\pi\sigma_0^2\rho a}{kM_0^2(ka)}\{\mathrm{ber}\,ka\,\mathrm{bei}'ka - \mathrm{bei}\,ka\,\mathrm{ber}'ka\}, \tag{26}$$

from (73), Chap. VIII,

$$= \frac{\pi\sigma_0^2\rho a}{k}\frac{M_1(ka)}{M_0(ka)}\cos(\theta_0 - \theta_1 + \tfrac{3}{4}\pi). \tag{27}$$

The alternating current resistance is $\mathbf{P}_1/I^2_{\text{r.m.s.}}$, this being either (26) divided by (20) or (27) divided by (21). Thus

$$R_e = \frac{\rho}{\pi a^2}\frac{ka}{2}\left\{\frac{\mathrm{ber}\,ka\,\mathrm{bei}'ka - \mathrm{bei}\,ka\,\mathrm{ber}'ka}{\mathrm{ber}'^2 ka + \mathrm{bei}'^2 ka}\right\} \tag{28}$$

$$= \frac{\rho}{\pi a^2}\frac{ka}{2}\frac{M_0(ka)}{M_1(ka)}\cos(\theta_0 - \theta_1 + \tfrac{3}{4}\pi) \tag{29}$$

per cm. length.

5. Skin effect in circular tubes

In § 1 the solution of equation (6) was given in the form

$$\sigma = A_1 J_0(kri^{\frac{3}{2}}) + B_1 K_0(kri^{\frac{1}{2}}).$$

Since the current density at the centre of a wire is finite, the second solution $K_0(kri^{\frac{1}{2}})$ is inadmissible. For a tube of inner radius b, $K_0(kbi^{\frac{1}{2}})$ is finite and the second solution can be retained. To determine the arbitrary constants A_1 and B_1, we require two equations. From above, when $\sigma = \sigma_0$, $r = a$ so

$$\sigma_0 = A_1 J_0(kai^{\frac{3}{2}}) + B_1 K_0(kai^{\frac{1}{2}}), \tag{30}$$

this being the first equation. At the inner surface of radius b, there is no e.m.f. induced by the magnetic field, so $\dfrac{dH}{dt} = 0$, and from (2), $\left(\dfrac{d\sigma}{dr}\right)_{r=b} = 0$. Accordingly, the second equation is

$$\left(\frac{d\sigma}{dr}\right)_{r=b} = ki^{\frac{1}{2}}\{A_1 iJ_0'(kbi^{\frac{3}{2}}) + B_1 K_0'(kbi^{\frac{1}{2}})\} = 0. \tag{31}$$

It is not proposed to pursue the problem any farther, so the evaluation of the ratio $R_e/R_{\text{d.c.}}$ for a circular tube, as given in example 11 at the end of this chapter, will be left to the reader.

6. Eddy current loss in core of solenoid

When a long solenoid carrying alternating current has a cylindrical metal core, the varying magnetic field induces currents therein, the core acting in the same manner as the secondary circuit of a transformer. If the solenoid is long compared with its radius, the magnetic field can be regarded as approximately parallel to the axis. The current distribution in the solenoid will be uniform, provided its fundamental natural frequency is appreciably greater than the frequency of the magnetizing current. The core will be assumed to consist of straight round wires or rods, usually of magnetic material, e.g. iron, mumetal, permalloy or the like, placed parallel to the axis of the solenoid. We shall now determine the heat loss due to the eddy currents induced in the core, the rods being insulated in effect. (See Fig. 24 A.)

Consider a ring 1 cm. long whose radius is r and thickness dr (Fig. 21 B). If H is the magnetic force and σ the current density at radius x, the total flux within the radius r is $\mu \int_0^r 2\pi x H\, dx$, where μ is the permeability of the core which we assume to be constant.† The e.m.f. induced in the ring is the rate of change of flux within radius r, so

$$dE = -\frac{\partial}{\partial t}\left\{\mu \int_0^r 2\pi x H\, dx\right\} = -2\pi\mu \int_0^r x \frac{\partial H}{\partial t}\, dx. \tag{32}$$

The resistance of the ring to circulating currents is $R = \dfrac{2\pi r\rho}{dr}$, whilst the current is $\sigma\, dr$. Thus

$$\sigma\, dr = \frac{dE}{R} = -2\pi\mu \int_0^r x \frac{\partial H}{\partial t}\, dx \Big/ \frac{2\pi r\rho}{dr},$$

or

$$\sigma = -\frac{\mu}{r\rho} \int_0^r x \frac{\partial H}{\partial t}\, dx. \tag{33}$$

The magnetizing force within the ring due to the eddy current $\sigma\, dr$ circulating in it is $-dH = 4\pi\sigma\, dr$, so

$$\sigma = -\frac{1}{4\pi}\frac{dH}{dr}. \tag{34}$$

† In practice μ usually varies considerably from point to point, during a cycle of an alternating current, so the results in the following sections are only approximate.

Equating (33) and (34) we obtain

$$r\frac{dH}{dr} = \frac{4\pi\mu}{\rho}\int_0^r x\frac{\partial H}{\partial t}\,dx. \tag{35}$$

If the flux varies sinusoidally $\frac{\partial H}{\partial t} = i\omega H$, so (35) becomes

$$r\frac{dH}{dr} = \frac{4\pi\mu\omega i}{\rho}\int_0^r xH\,dx. \tag{36}$$

Differentiating (36) with respect to r and inserting the limits, we have

$$r\frac{d^2H}{dr^2}+\frac{dH}{dr} = \frac{4\pi\mu\omega i}{\rho}Hr$$

or

$$\frac{d^2H}{dr^2}+\frac{1}{r}\frac{dH}{dr}-im^2H = 0, \tag{37}$$

where $m^2 = 4\pi\mu\omega/\rho$. From (4), Chap. VIII, the solution of (37)† is

$$H = A_1 J_0(mri^{\frac{3}{2}}), \tag{38}$$

the second solution being inadmissible, since H is not infinite on the axis of the core wire.

If H_0 is the r.m.s. field-strength at the surface of the wire due to the current in the solenoid, we have on substituting $r = a$,

$$H_0 = A_1 J_0(mai^{\frac{3}{2}}),$$

so that

$$H = \frac{H_0 J_0(mri^{\frac{3}{2}})}{J_0(mai^{\frac{3}{2}})}. \tag{39}$$

From (34) and (39)

$$\sigma = -\frac{H_0}{4\pi J_0(mai^{\frac{3}{2}})}\frac{d}{dr}J_0(mri^{\frac{3}{2}})$$

$$= \frac{i^{\frac{3}{2}}H_0 m}{4\pi}\frac{J_1(mri^{\frac{3}{2}})}{J_0(mai^{\frac{3}{2}})}$$

$$= \frac{H_0 m}{4\pi}\frac{M_1(mr)}{M_0(ma)}e^{i\{\theta_1(mr)-\theta_0(ma)+\frac{3}{4}\pi\}}. \tag{40}$$

The mean square value of the current in the ring at radius r is, therefore,

$$\sigma^2_{\text{r.m.s.}}\,dr^2 = \frac{H_0^2 m^2}{16\pi^2}\frac{M_1^2(mr)}{M_0^2(ma)}\,dr^2. \tag{41}$$

The heat loss in the ring is $\left(\frac{\rho 2\pi r}{dr}\right)\sigma^2_{\text{r.m.s.}}\,dr^2 = \frac{H_0^2 m^2\rho}{8\pi}\frac{M_1^2(mr)}{M_0^2(ma)}r\,dr$, so

† The form of (37) is identical with that of (5).

the total loss in 1 cm. length is

$$\mathbf{P}_1 = \frac{H_0^2 m^2\rho}{8\pi M_0^2(ma)} \int\limits_0^a M_1^2(mr)r\,dr. \tag{42}$$

Using the result in example 21, Chap. VIII, we find that

$$\mathbf{P}_1 = \frac{H_0^2\rho ma}{8\pi}\frac{M_1(ma)}{M_0(ma)}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi), \tag{43}$$

or
$$\mathbf{P}_1 = \frac{H_0^2\rho ma}{8\pi}\left\{\frac{\operatorname{ber} ma \operatorname{ber}' ma+\operatorname{bei} ma \operatorname{bei}' ma}{\operatorname{ber}^2 ma+\operatorname{bei}^2 ma}\right\}. \tag{44}$$

This is the loss per cm. length of the wire in ergs per second. The loss per cubic centimetre is, therefore,

$$\mathbf{P}_{\text{c.c.}} = \frac{H_0^2\rho m}{8\pi^2 a}\frac{M_1(ma)}{M_0(ma)}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi) \text{ ergs per second.} \tag{45}$$

Taking $H_0 = \dfrac{4\pi n I_{\text{r.m.s.}}}{10l}$, as a first approximation for a long solenoid or an anchor ring (toroid), where n is the total turns uniformly distributed over a mean length l, formula (45) can be written

$$\mathbf{P} = \frac{4\pi n^2\mu\omega}{l^2} I_{\text{r.m.s.}}^2 \mathbf{W}(ma)\times 10^{-9} \text{ watts per cm.}^3, \tag{46}$$

where $\mathbf{W}(ma) = \dfrac{2}{ma}\dfrac{M_1(ma)}{M_0(ma)}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)$, and the current is in amperes. When $ma > 10$, $\mathbf{W}(ma) \doteqdot \dfrac{\sqrt{2}}{ma}$.

7. Condition for maximum heat loss in core

The loss function $\mathbf{W}(ma)$ is plotted in Fig. 24, curve 1, and it has a maximum value in the neighbourhood of $ma = 2{\cdot}5$. The requisite condition for maximum loss can be obtained for a given material by varying either the frequency or the diameter of the core rods. The latter procedure must be adopted in practice since the frequency is fixed. The magnitude of the maximum loss is given by

$$\mathbf{P}_{\max} = \frac{4{\cdot}62 n^2\mu\omega I_{\text{r.m.s.}}^2}{l^2}\times 10^{-9} \text{ watts per cm.}^3 \text{ of core.} \tag{47}$$

The existence of a maximum loss can be explained in the following way. If the core consists of extremely thin wires, the eddy currents at moderate frequencies will be too small to cause appreciable loss. On the other hand, if the core consists of a few very thick rods, the

eddy currents will be relatively weak at the central part of the wire. This can be seen immediately from (41), since, for a given radius r, the value of $M_0(ma)$ increases with increase in a, the outer radius of the wire. Hence the eddy currents at a given radius decrease with increase in the size of the wire. It follows that there is a certain diameter of wire for which the heat generated in a core of definite volume is a maximum.

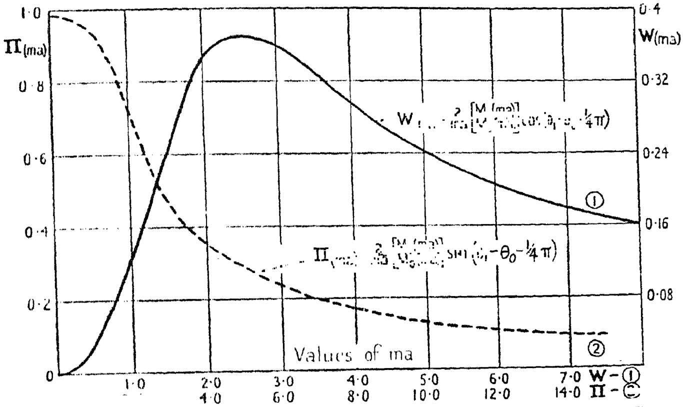


FIG. 24. The loss function $W(ma)$ and the penetration function $\Pi(ma)$.

8. Effective resistance of solenoid due to core loss (R_e)

We have $P = I^2_{\text{r.m.s.}} R_e$, where P is the total loss due to the core.

From (46), $P = \dfrac{4\pi n^2\mu\omega}{l^2} I^2_{\text{r.m.s.}} V\mathrm{W}(ma)\times 10^{-9}$, where V is the volume of the core, so

$$R_e = \frac{4\pi n^2\mu\omega V\mathrm{W}(ma)}{l^2}\times 10^{-9} \text{ ohms.} \tag{48}$$

The maximum possible value of R_e is from (47)

$$R_{e,\max} = \frac{4\cdot 62 n^2\mu\omega V}{l^2}\times 10^{-9} \text{ ohms.} \tag{49}$$

9. Effective inductance of toroid due to core (L_e)

The magnetizing force in phase with the flux at the surface of a core wire is $H_0\sin(\theta_1-\theta_0-\frac{1}{4}\pi)$, so the flux therein is

$$2\pi\mu\int_0^a Hr\,dr = \frac{2\pi\mu H_0}{J_0(mai^{\frac{3}{2}})}\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi)\int_0^a J_0(mri^{\frac{3}{2}})r\,dr,$$

or $$\Phi = \frac{2\pi a^2 \mu H_0}{J_0(mai^{\frac{3}{2}})} \frac{J_1(mai^{\frac{3}{2}})}{ma} \sin(\theta_1 - \theta_0 - \tfrac{1}{4}\pi) e^{-\frac{1}{4}\pi i}$$

$$= 2\pi a^2 \mu H_0 \frac{1}{ma} \frac{M_1(ma)}{M_0(ma)} \sin(\theta_1 - \theta_0 - \tfrac{1}{4}\pi) e^{i\{\theta_1(ma) - \theta_0(ma) - \frac{1}{4}\pi\}}.$$

Thus $$\Phi_{\max} = \pi a^2 \mu H_{\max} \left[\frac{2}{ma} \frac{M_1(ma)}{M_0(ma)}\right] \sin(\theta_1 - \theta_0 - \tfrac{1}{4}\pi).$$

Using the formula for the magnetizing force within a toroidal winding, whose radial width is small compared with the mean radius of the toroid, namely, $H_{\max} = 4nI_{\max}/d_m$, we obtain

$$\Phi_{\max} = 4\pi a^2 n \mu I_{\max} \mathbf{\Pi}(ma)/d_m, \tag{50}$$

where d_m is the mean diameter of the ring, $I_{\max}$ is in absolute e.m. units, and $\mathbf{\Pi}(ma) = \dfrac{2}{ma} \dfrac{M_1(ma)}{M_0(ma)} \sin(\theta_1 - \theta_0 - \tfrac{1}{4}\pi)$. The penetration function $\mathbf{\Pi}(ma)$ is plotted in Fig. 24, curve 2. It will be seen that

$$\mathbf{W}(ma)/\mathbf{\Pi}(ma) = \cot(\theta_1 - \theta_0 - \tfrac{1}{4}\pi),$$

and when $ma > 10$ this ratio is approximately 1.

The total flux interlinkage for p core wires and n turns on the toroid, per unit current absolute, is

$$L_e = pn\Phi_{\max} \times 10^{-9}/I_{\max}$$

$$= \frac{4n^2 A \mu \mathbf{\Pi}(ma) \times 10^{-9}}{d_m} \text{ henrys}, \tag{51}$$

where A is the total cross-sectional area of the core. The effective inductance can also be expressed as

$$L_e = \frac{4\pi n^2 A \mu_e \times 10^{-9}}{\pi d_m} = \frac{4n^2 A \mu_e \times 10^{-9}}{d_m}, \tag{52}$$

where μ_e is the effective permeability of the core.

It should be remarked that (52) gives L_e due to the metal of the core only. Since the cross-section of the toroid is partly non-magnetic (insulation and air space), the corresponding inductance must be added to (52) to obtain the total inductance. Thus

$$L_e \text{ (total)} = \frac{4n^2}{d_m} (A\mu_e + A_0) \times 10^{-9}, \tag{52 a}$$

where A_0 is the area of the non-magnetic portion of the section. From (51) and (52)

$$\mu_e = \mu \mathbf{\Pi}(ma). \tag{53}$$

When $ma > 10$, $\mathbf{\Pi}(ma) \doteqdot \sqrt{2}/ma$, so $\mu_e \doteqdot \sqrt{2}\mu/ma$, (54)

whilst† when $ma \to 0$ at low frequencies $\mathbf{\Pi}(ma) \to 1$ and from (53) $\mu_e \to \mu$.

10. Application of preceding analysis to electric furnaces of the eddy current type

The above analysis can be applied to electric furnaces where the core is melted due to heat caused by eddy currents. The solenoid or inductor consists of a few turns of water-cooled tube placed round a crucible of suitable non-conducting material. The charge to the

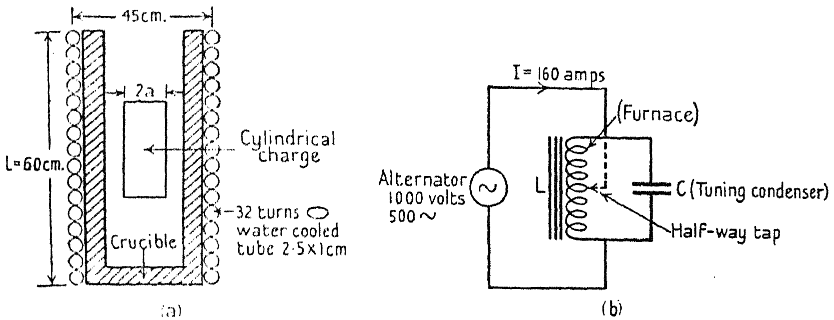


FIG. 24 A and B

crucible may be in the form of rods or lumps, the former being preferred. They should be placed parallel to the axis of the inductor. The frequency of operation is governed by economic considerations respecting the general design of the plant and the cost of constructional materials. To secure efficient working it is essential to operate the circuit at unity power factor. This is accomplished by tapping the coil and varying the tuning condensers (Fig. 24 B). After the furnace is started, the permeability and resistivity of the charge both increase with the temperature. Meanwhile, by a relay system, condensers are removed to preserve unity power factor. At about 500° C. the half-way tap on the coil is used (Fig. 24 B), and the condensers altered accordingly. At the recalescence point (about 760° C. for iron) the permeability suddenly falls to unity. At a higher temperature, governed by the carbon content, the iron becomes magnetic, but returns to the non-magnetic state when molten. In this state the effective permeability is much less than unity owing to the eddy currents in the molten mass of large diameter. This means that the

† See Tables 14 and 15.

eddy currents reduce the flux to such an extent that the arrangement resembles a power transformer with a short-circuited secondary winding.

In view of the variations in μ and ρ† described above, it is rather puzzling to fix the precise point where the optimum criterion $ma = 2{\cdot}5$ can be applied. In practice the dimensions of the metal rods can easily be varied to ascertain the optimum diameter. There is a definite optimum, which for a 2,000 ∼ furnace is about 1·3 cm. when melting iron. From the formula we find $\rho/\mu = 43{,}000$, where ρ is in absolute electromagnetic units. This value indicates that the optimum criterion is applicable at high temperatures. In the molten state the core diameter is the internal diameter of the crucible, this being many times that of the cold charge. Consequently conditions must be adjusted so that the iron melts readily and can easily be kept in that state. In general it is found when metal at the bottom of the crucible melts, that the remainder soon does likewise. It is then possible to feed thin scrap sheet to the molten metal, which soon causes the scrap to melt. Starting from cold the thin sheet might not melt at all.

EXAMPLES

1. The resistance of a long straight copper wire of circular section at 500 kilocycles is 5 times its value to direct current. Find the radius of the wire, and plot curves showing the current density and phase angle at any radius. ρ for copper is 1,600 abs. e.m. units. [$a = 0{\cdot}86$ mm.]

2. In § 1 instead of solving the equations for σ, solve for H the magnetic force at any radius r. Plot H and r for a copper wire 1 mm. radius at 10^6 ∼.

$$\left[H = \frac{4\pi\sigma_0}{k}\frac{M_1(kr)}{M_0(ka)}e^{i\{\theta_1(kr)-\theta_0(ka)-\frac{3}{4}\pi\}}.\right]$$

3. In example 2 what is the magnetic force at any point outside the wire? What is H when the frequency tends to zero?

$$\left[H = \frac{2I}{r} = \frac{4\pi\sigma_0 a}{kr}\frac{M_1(ka)}{M_0(ka)}e^{i\{\theta_1(ka)-\theta_0(ka)-\frac{3}{4}\pi\}};\ \frac{2\pi a^2\sigma_0}{r}.\right]$$

4. A solenoid 45 cm. long 15 cm. mean diameter has 20 turns. The core consists of 100 metal rods each 0·6 cm. diameter and 30 cm. long placed centrally parallel to the axis. Find (1) the watt loss due to the core, (2) the effective resistance due to the core, (3) the effective permeability of the core at $\omega = 4{,}000\pi$. Take $\mu = 500$, $\rho = 9\times 10^3$ abs. e.m. units, r.m.s. current $= 50$ amperes.

[(1) 1,700 watts, (2) 0·68 ohm, (3) $\mu_e = 25{\cdot}2$.]

† ρ for steel at the melting-point is about $1{\cdot}8\times 10^5$ absolute units or some twenty times its value at ordinary temperatures.

5. In example 4, if the core length and mass remain unaltered, what size of wire gives the maximum initial heat loss before the temperature rises appreciably? What is the heat loss then?

[0·54 mm. diameter; 7·3 times that in example 4.]

6. A uniformly wound toroidal ring 10 cm. mean radius has 500 turns and the cross-sectional area of the core is 5 sq. cm. If the ring resonates with a 2·5 μF condenser at 1,000 $\sim$, what is the diameter of the core wire? Take the normal permeability of the core wire as 800 and its resistivity 10^4 abs. e.m. units. Calculate the effective permeability and the core loss per ampere (r.m.s.).

[3·54 mm. diameter; $L_e = 0{\cdot}01$ henry; $\mu_e = 40$; watt loss per ampere 20π.]

7. If the solenoid in example 4 were fitted with a crucible of 10 cm. internal diameter filled with a solid charge of the same diameter and length 30 cm., calculate the heat loss when the r.m.s. current is 200 amperes, $\mu = 1$, and $\rho = 10^5$ abs. e.m. units. What is the effective resistance and the effective permeability of the charge?

[$P = 570$ watts; $R_e = 1{\cdot}4\times 10^{-2}$ ohm; $\mu_e = 0{\cdot}23$.]

8. If under certain conditions the permeability of a magnetic material follows a law $\mu = 500+5\times 10^{-6}t^3$, where t is the temperature in °C., plot a curve showing the total loss at various temperatures up to 750° C., when the core is used in a solenoid whose dimensions are given in example 4. The core consists of wires 0·4 mm. diameter whose total cross-section and volume are the same as in example 4. Assume the resistivity to follow the law $\rho = 9\times 10^3(1+0{\cdot}0063t)$; $\omega = 4{,}000\pi$, r.m.s. current 50 amperes.

9. A toroidal winding carrying an alternating current of angular frequency ω has a core of iron wires. Show that the effective permeability of the core is unity when $\omega \doteqdot \mu\rho/2\pi a^2$, where a is the radius of the core wires.

10. Show that the penetration function, i.e. $\Pi(ma)$, represents the ratio of the mean flux density in a core wire in example 9 to that when the flux is uniformly distributed over the cross-section.

11. The ratio of the alternating current to the direct current resistance of an isolated straight tubular conductor is the real part of [37]

$$\psi = \tfrac{1}{2}ima\left(1-\frac{b^2}{a^2}\right)\left\{\frac{(\operatorname{ber} ma+i\operatorname{bei} ma)+\varphi(\operatorname{ker} ma+i\operatorname{kei} ma)}{\operatorname{ber}' ma+i\operatorname{bei}' ma+\varphi(\operatorname{ker}' ma+i\operatorname{kei}' ma)}\right\},$$

where $$\varphi = -\left[\frac{\operatorname{ber}' mb+i\operatorname{bei}' mb}{\operatorname{ker}' mb+i\operatorname{kei}' mb}\right].$$

(*a*) Write ψ in terms of J_0, J_0', K_0, K_0'.

(*b*) Evaluate the real part of ψ when the frequency is 550 cycles per second, outer radius $a = 1{\cdot}5$ cm., inner radius $b = 0{\cdot}9$ cm., $\rho = 1{,}737$ absolute electromagnetic units, $m^2 = 4\pi\omega/\rho$, $\omega = 2\pi\times 550$.

$$\left[(a)\ \psi = \left(\frac{1-i}{2\sqrt{2}}\right)ma\left(1-\frac{b^2}{a^2}\right)\left\{\frac{J_0(mai^{\frac{3}{2}})K_0'(mbi^{\frac{1}{2}})-iJ_0'(mbi^{\frac{3}{2}})K_0(mai^{\frac{1}{2}})}{J_0'(mai^{\frac{3}{2}})K_0'(mbi^{\frac{1}{2}})-J_0'(mbi^{\frac{3}{2}})K_0'(mai^{\frac{1}{2}})}\right\};\right.$$

$$\left.(b)\ \frac{R_e}{R_{\text{d.c.}}} = 1{\cdot}8.\ \text{See paper cited in preface for polar form of } \psi.\right]$$

MISCELLANEOUS EXAMPLES

1. Show that $J_2(z)I_3(z)+J_3(z)I_2(z) = J_2(z)I_1(z)-J_1(z)I_2(z)$.

2. Show that $zJ_1(z) = 4J_2(z)-zJ_3(z) = 4J_2(z)-8J_4(z)+zJ_5(z)$

$$= 4\sum_{r=1}^{m}(-1)^{r-1}rJ_{2r}(z)+(-1)^m zJ_{2m+1}(z) \text{ [93]}.$$

3. Show that $zJ_{2m+1}(z)$ tends to zero as $m \to \infty$, and hence from example 2

that $zJ_1(z) = 4\sum_{n=1}^{\infty}(-1)^{n-1}nJ_{2n}(z)$.

4. Using the recurrence formulae

$$(a)\ \frac{d}{dz}\{z^\nu J_\nu(z)\} = z^\nu J_{\nu-1}(z), \qquad (b)\ \frac{d}{dz}\left\{\frac{J_{\nu-1}(z)}{z^{\nu-1}}\right\} = -\frac{J_\nu(z)}{z^{\nu-1}}$$

substitute for $J_{\nu-1}(z)$ from (a) in (b), and thence reproduce Bessel's general equation in which y is replaced by $J_\nu(z)$.

5. Prove $J_n(-z) = (-1)^n J_n(z)$ when n is a positive integer: hence show that if $ikI_1(mzi)+lI_0(mzi) = 0$, $kJ_1(mz)-lJ_0(mz) = 0$.

6. When z_1 and $z_2 > 10$, show that

$$(a)\quad \int_{z_2}^{z_1} K_0(z)I_0(z)\,dz \doteqdot \tfrac{1}{2}\log\frac{z_1}{z_2};$$

$$(b)\quad \int_{z_2}^{z_1} K_0(z)I_0(z)z\,dz \doteqdot \tfrac{1}{2}(z_1-z_2).$$

[Use asymptotic expansions.]

7. Solve $\dfrac{d^2y}{dz^2}+\left(\dfrac{2p+1}{z}\right)\dfrac{dy}{dz}+\left(1-\dfrac{\nu^2-p^2}{z^2}\right)y = 0.$

$$\left[y = \frac{1}{z^p}\{A_1J_n(z)+B_1Y_n(z)\} \quad \text{or} \quad y = \frac{1}{z^p}\{A_1J_\nu(z)+B_1J_{-\nu}(z)\},\right.$$

according as ν is integral or fractional. Substitute $y = vz^{-p}$.]

8. Solve $\dfrac{d^2y}{dz^2}+\left(k^2-\dfrac{n^2-\frac{1}{4}}{z^2}\right)y = 0.$

[$y = z^{\frac{1}{2}}\{A_1J_n(kz)+B_1Y_n(kz)\}$, if n is integral. Substitute $y = vz^{\frac{1}{2}}$.]

9. Solve $\dfrac{d^2y}{dz^2}+2k^2zy = 0.$

$$\left[y = z^{\frac{1}{2}}\left\{A_1J_{\frac{1}{3}}\left(\frac{2^{\frac{3}{2}}kz^{\frac{3}{2}}}{3}\right)+B_1J_{-\frac{1}{3}}\left(\frac{2^{\frac{3}{2}}kz^{\frac{3}{2}}}{3}\right)\right\}.\ \text{See example 46, Chap. II.}\right]$$

10. Show that [93] $y = F(\alpha;\gamma;z) = 1+\dfrac{\alpha z}{1!\gamma}+\dfrac{\alpha(\alpha+1)}{2!\gamma(\gamma+1)}z^2+\ldots$ is a solution

of $z\dfrac{d^2y}{dz^2}+(\gamma-z)\dfrac{dy}{dz}-\alpha y = 0.$

11. Show that [93] $y = e^{-iz}F[\frac{1}{2}(1+ia);1;2iz]$ is a solution of

$$z\frac{d^2y}{dz^2}+\frac{dy}{dz}+(z+a)y = 0,$$

where F is defined in 10 above.

12. Solve $\dfrac{d^4y}{dz^4} = k^4 y$ in terms of Bessel Functions. What is the solution in terms of circular and hyperbolic functions?

[$y = z^{\frac{1}{2}}\{A_1 J_{\frac{1}{2}}(kz) + B_1 J_{-\frac{1}{2}}(kz) + C_1 I_{\frac{1}{2}}(kz) + D_1 I_{-\frac{1}{2}}(kz)\}$. Use the operator $D = d/dz$ and factorize to solve in terms of circular and hyperbolic functions.]

13. Solve $\dfrac{d^4y}{dz^4} + k^4 y = 0$ in terms of Bessel functions.

[Use $D = d/dz$ and factorize.]

14. From Bessel's differential equation we have

$$zJ_0''(kz) + \frac{1}{k}J_0'(kz) + zJ_0(kz) = 0.$$

By integrating this equation show that

$$\int_0^a zJ_0(kz)\,dz = -\frac{a}{k}J_0'(ka) = \frac{a}{k}J_1(ka).$$

15. Find a solution [77] of

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} - \frac{1}{\beta^2}\frac{\partial\phi}{\partial t} + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\chi^2}\right\} = 0, \qquad (1)$$

by assuming $\phi = e^{-\alpha^2 t}u\,S_n(\theta, \chi)$, where u is a function of r only, and $S_n(\theta, \chi)$ is a function of θ (see Fig. 8) and χ (longitude) which satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\chi^2} + n(n+1)S_n = 0,$$

n being zero or a positive integer.

[$\phi = e^{-\alpha^2 t}r^{-\frac{1}{2}}S_n(\theta, \chi)\{A_1 J_{n+\frac{1}{2}}(kr) + B_1 J_{-n-\frac{1}{2}}(kr)\}$, where $k = \alpha/\beta$. Obtain the value of the quantity in brackets in equation (1) from the second equation, then substitute $u = vr^{-\frac{1}{2}}$. The original equation is associated with the flow of heat in a sphere raised to a certain temperature and allowed to cool under definite conditions. The second equation pertains to spherical harmonics (S_n), which are of importance in acoustical work and in the theory of elasticity (see § 7, Chap. II and [91 a]).]

16. Show that $J_n(z) = \dfrac{(\frac{1}{2}z)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\displaystyle\int_{-1}^{+1} e^{\pm izt}(1-t^2)^{n-\frac{1}{2}}\,dt$ (n a positive integer).

17. Show that, when n is a positive integer,

$$\frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{(\frac{1}{2}z)^n}I_n(z) = \int_0^\pi e^{\pm z\cos\theta}\sin^{2n}\theta\,d\theta = \int_0^\pi \cosh(z\cos\theta)\sin^{2n}\theta\,d\theta$$

$$= \int_{-1}^{+1} e^{\pm zt}(1-t^2)^{n-\frac{1}{2}}\,dt.$$

18. Show that $J_n''(z) = \left(\dfrac{n^2}{z^2} - 1\right)J_n(z)$ when $J_n(z)$ is a maximum or a minimum ($z \neq 0$).

19. Given that $J_0(z) = \frac{2}{\pi}\int\limits_1^\infty \frac{\sin(kz)\,dk}{\sqrt{(k^2-1)}}$, show that $J_0(z) = \frac{2}{\pi}\int\limits_0^\infty \sin(z\cosh\theta)\,d\theta$.

20. Show that (a) $I_\nu(ze^{im\pi}) = e^{m\nu\pi i}I_\nu(z)$,

(b) $K_\nu(ze^{im\pi}) = e^{-m\nu\pi i}K_\nu(z) - i\pi\frac{\sin m\nu\pi}{\sin\nu\pi}I_\nu(z)$.

21. When $z \to 0$, show that

(a) $z^{-\nu}I_\nu(z) \to 1/2^\nu\Gamma(\nu+1)$; (b) $z^{-\nu+1}I'_\nu(z) \to \nu/2^\nu\Gamma(1+\nu)$;

(c) $z^\nu I_{-\nu}(z) \to 2^\nu/\Gamma(1-\nu)$; (d) $z^{\nu+1}I'_{-\nu}(z) \to -2^\nu\nu/\Gamma(1-\nu)$;

(e) $J_\nu(z)/I_\nu(z) \to 1$.

22. Evaluate the following integral which occurs in determining the distribution of sound from a vibrating disk [6]:

$$\int\limits_0^a I_0(k_1 r)J_0(kr\sin\phi)r\,dr.$$

$$\left[\frac{a}{k_1^2+k^2\sin^2\phi}\{k_1 J_0(ka\sin\phi)I_1(k_1 a)+k\sin\phi I_0(k_1 a)J_1(ka\sin\phi)\}.\right]$$

23. Evaluate the following integral which occurs in determining the distribution of sound from a vibrating disk:

$$\int\limits_0^a K_0(k_1 r)J_0(kr\sin\phi)r\,dr.$$

$$\left[\frac{a}{k_1^2+k^2\sin^2\phi}\{-k_1 J_0(ka\sin\phi)K_1(k_1 a)+k\sin\phi K_0(k_1 a)J_1(ka\sin\phi)\}+\frac{1}{k_1^2+k^2\sin^2\phi}.\right]$$

24. Show that $\frac{d}{dz}\{\log J_0(z)\} \doteqdot -\frac{1}{2}z$ when $z \ll 1$: also show that $-\frac{1}{2}z(1+\frac{1}{8}z^2)$ is a closer approximation.

25. Prove that $\int\limits^z z^{n+1}J_n(z)\,dz = nz^nJ_n(z)+\frac{z^{n+1}}{2}[J_{n+1}(z)-J_{n-1}(z)]$.

26. Prove that $\Gamma(z) = \int\limits_0^\infty e^{-t}t^{z-1}\,dt$, provided $R(z) > 0$.

$\Big[$Write $\int\limits_0^n (1-t/n)^n t^{z-1}\,dt = \int\limits_0^n \frac{(1-t/n)^n}{z}\,d(t^z)$.

Integrate progressively by parts and discuss limits.$\Big]$

27. Prove that $\frac{K_\nu(z)}{K_{\nu+1}(z)-K_{\nu-1}(z)} = \frac{I_\nu(z)}{I_{\nu-1}(z)-I_{\nu+1}(z)} = \frac{J_\nu(z)}{J_{\nu+1}(z)+J_{\nu-1}(z)}$.

28. Prove that $J_\nu^2(z) = \frac{1}{2\nu}\int\limits^z \{J_{\nu-1}^2(z)-J_{\nu+1}^2(z)\}z\,dz$.

29. Show that $zJ_0^2(z) = \frac{1}{2}\frac{d}{dz}\{z^2[J_0^2(z)+J_1^2(z)]\}$

30. Show that (*a*) $J_{\frac{1}{2}}(z)J_{-\frac{1}{2}}(z) = \dfrac{\sin 2z}{\pi z}$;

$$(b)\ J^2_{\frac{1}{2}}(z)+J^2_{-\frac{1}{2}}(z) = \frac{2}{\pi z};$$

$$(c)\ [J_{\frac{1}{2}}(z)+J_{-\frac{1}{2}}(z)]^2 = \frac{2}{\pi z}(1+\sin 2z);$$

$$(d)\ J^2_{\frac{3}{2}}(z)+J^2_{-\frac{3}{2}}(z) = \frac{2}{\pi z}\left(1+\frac{1}{z^2}\right).$$

31. When z is real show that (*a*) $\lim\limits_{z\to\infty} |Y_0(zi^{\frac{3}{2}})| = \infty$;

$$(b)\ \lim_{z\to\infty} |K_0(zi^{\frac{1}{2}})| = 0.$$

[Use asymptotic formulae.]

32. Plot $z\dfrac{J_0(z)}{J_1(z)}$ from $z = -6$ to $+6$.

33. Solve $\dfrac{\partial^2\phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\phi}{\partial r}+\dfrac{\partial^2\phi}{\partial z^2}-2\beta\dfrac{\partial\phi}{\partial z} = 0$. [Only one solution required.]

[$\phi = J_0(kr)e^{\beta z}[A_1\cosh\{z\sqrt{(k^2+\beta^2)}\}+B_1\sinh\{z\sqrt{(k^2+\beta^2)}\}]$, where $k^2 = \alpha^2-2\alpha\beta$. Put $\phi = \chi e^{\alpha z}$, where χ is a function of r but not of z.]

34. Show that $\operatorname{ber}_1 z = \dfrac{1}{\sqrt{2}}(\operatorname{ber}'z-\operatorname{bei}'z)$,

$$\operatorname{bei}_1 z = \frac{1}{\sqrt{2}}(\operatorname{ber}'z+\operatorname{bei}'z).$$

35. Show that $\lim\limits_{z\to 0} z[lJ_\nu(kz)K'_\nu(lz)-kK_\nu(lz)J'_\nu(kz)] = -(k/l)^\nu$. $R(\nu) > -1$.

36. Show that $J'_\nu(zi^{\frac{3}{2}}) = \dfrac{1}{\sqrt{2}}\{(\operatorname{ber}'_\nu z-\operatorname{bei}'_\nu z)+i(\operatorname{ber}'_\nu z+\operatorname{bei}'_\nu z)\}$.

37. Prove that

$$(a)\ \int_0^z z(\operatorname{ber}^2 z-\operatorname{bei}^2 z)\,dz = \tfrac{1}{2}z^2\{(\operatorname{ber}^2 z-\operatorname{bei}^2 z)-2\operatorname{ber}'z\operatorname{bei}'z\}$$
$$= \tfrac{1}{2}z^2\{(\operatorname{ber}^2 z-\operatorname{bei}^2 z)+(\operatorname{ber}_1^2 z-\operatorname{bei}_1^2 z)\};$$

$$(b)\ \int_0^z z\operatorname{ber} z\operatorname{bei} z\,dz = \tfrac{1}{4}z^2\{2\operatorname{ber} z\operatorname{bei} z+(\operatorname{ber}'^2 z-\operatorname{bei}'^2 z)\}$$
$$= \tfrac{1}{2}z^2(\operatorname{ber} z\operatorname{bei} z+\operatorname{ber}_1 z\operatorname{bei}_1 z).$$

[Integrate $zJ_0^2(zi^{\frac{3}{2}})\,dz$ then equate real and imaginary parts.]

38. Find an infinite series for $\operatorname{ber}^2 z+\operatorname{bei}^2 z$ given that

$$J_\nu(az)J_\nu(azi) = \sum_{r=0}^{\infty}\frac{(\frac{1}{2}azi^{\frac{1}{2}})^{2\nu+4r}}{r!\,\Gamma(\nu+r+1)\Gamma(\nu+2r+1)}.$$

[$\operatorname{ber}^2 z+\operatorname{bei}^2 z = 1+\dfrac{(\frac{1}{2}z)^4}{2!}+\dfrac{(\frac{1}{2}z)^8}{(2!)^2 4!}+\dfrac{(\frac{1}{2}z)^{12}}{(3!)^2 6!}+\dots$. Put $\nu = 0$, $a = i^{\frac{3}{2}}$, and $i^{\frac{3}{2}} = i^4 i^{-\frac{5}{2}}$, but not $i^{\frac{3}{2}} = i^{-\frac{5}{2}}$ since the angles are different.]

39. Prove that (*a*) $z\operatorname{ber}'_\nu z = \nu\operatorname{ber}_\nu z+\dfrac{z}{\sqrt{2}}(\operatorname{ber}_{\nu+1} z+\operatorname{bei}_{\nu+1} z)$;

$$(b)\ z\operatorname{bei}'_\nu z = \nu\operatorname{bei}_\nu z-\frac{z}{\sqrt{2}}(\operatorname{ber}_{\nu+1} z-\operatorname{bei}_{\nu+1} z);$$

$$(c)\ z\,\mathrm{ber}'_\nu z = -\nu\,\mathrm{ber}_\nu z - \frac{z}{\sqrt{2}}(\mathrm{ber}_{\nu-1}z + \mathrm{bei}_{\nu-1}z);$$

$$(d)\ z\,\mathrm{bei}'_\nu z = -\nu\,\mathrm{ber}_\nu z + \frac{z}{\sqrt{2}}(\mathrm{ber}_{\nu-1}z - \mathrm{bei}_{\nu-1}z).$$

[Use recurrence formulae for $J_\nu(zi^{\frac{3}{2}})$.]

40. In § 2, Chap. IX, find the current density at any radius in terms of the total current in the wire.

41. Evaluate $\int\limits_0^a (\mathrm{ber}_n^2 z + \mathrm{bei}_n^2 z)z\,dz$. [$a(\mathrm{ber}_n a\,\mathrm{bei}'_n a - \mathrm{bei}_n a\,\mathrm{ber}'_n a)$.]

42. Evaluate $\int\limits_0^a (\mathrm{ker}_n^2 z + \mathrm{kei}_n^2 z)z\,dz$. [$a(\mathrm{ker}_n a\,\mathrm{kei}'_n a - \mathrm{kei}_n a\,\mathrm{ker}'_n a)$.]

43. Show that, when $k^2 = l^2$ and $R(\nu) > -1$,

$$-\int\limits_0^z I_\nu(kz)K_\nu(lz)z\,dz = \frac{\nu}{2l^2} + \tfrac{1}{2}z^2\left\{I'_\nu(lz)K'_\nu(lz) - I_\nu(lz)K_\nu(lz)\left(1 + \frac{\nu^2}{l^2z^2}\right)\right\}.$$

Verify that the value of this expression is zero when $z \to 0$.

[Interchange k and l in formula (133), p. 166, put $k = l+\epsilon$ so that $l^2 - k^2 = -(2l+\epsilon)\epsilon$, where ϵ is a small quantity. Expand $I_\nu\{z(l+\epsilon)\}$, $I'_\nu\{z(l+\epsilon)\}$ by Taylor's theorem and use the first two terms in each case. From example 60 obtain the value of $I_\nu(lz)K'_\nu(lz) - I'_\nu(lz)K_\nu(lz) = -1/lz$, and make $\epsilon \to 0$. To obtain the second term on the right-hand side of the integral in the form given, use the differential equation for $I_\nu(lz)$, i.e. $y'' + \frac{1}{x}y' - \left(1+\frac{\nu^2}{x^2}\right)y = 0$; put $y = I_\nu(lz)$ and $lz = x$.]

44. Show that (a) $\lim\limits_{z\to\infty} \dfrac{J'_0(zi)}{J_0(zi)} = -i$;

(b) a maximum value of $\dfrac{z(1-z^2)I_1(z)}{I_0(z)}$ occurs when $z \doteq 0{\cdot}68$.

[Expand and use a method of successive approximation.]

45. When z is extremely large and the imaginary part positive, show that

$$\frac{J'_0(z)}{J_0(z)} = \frac{d}{dz}\log J_0(z) \doteq \tan(\tfrac{1}{4}\pi - z) \doteq -i.$$

46. Solve $z^2\dfrac{d^2u}{dz^2} + z\dfrac{du}{dz} - \nu^2(1-z^2)u = 0$. [$u = A_1 J_\nu(\nu z) + B_1 Y_\nu(\nu z)$.]

47. When z is large enough, verify that to a first approximation

(a) $\mathrm{ber}\,z = \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\cos\left(\dfrac{z}{\sqrt{2}} - \tfrac{1}{8}\pi\right)$; (b) $\mathrm{bei}\,z = \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\sin\left(\dfrac{z}{\sqrt{2}} - \tfrac{1}{8}\pi\right)$;

(c) $M_0(z) = \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}$; (d) $\log_{10}\{M_0(z)\sqrt{z}\} = 0{\cdot}307z - 0{\cdot}399$;

(e) $\theta_0(z) = (40{\cdot}51z - 22{\cdot}5)^\circ$; (f) $\mathrm{ber}_1 z = \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\cos\left(\dfrac{z}{\sqrt{2}} + \tfrac{3}{8}\pi\right)$;

(g) $\mathrm{bei}_1 z = \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\sin(z/\sqrt{2} + \tfrac{3}{8}\pi)$; (h) $M_1(z) = \dfrac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}$;

(i) $\log_{10}\{M_1(z)\sqrt{z}\} = 0{\cdot}307z - 0{\cdot}399$; (j) $\theta_1(z) = (40{\cdot}51z + 67{\cdot}5)^\circ$.

48. When z is large enough, verify that to a first approximation

(a) $\ker z = \sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\cos\left(\frac{z}{\sqrt{2}}+\frac{1}{8}\pi\right)$;

(b) $\operatorname{kei} z = -\sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\sin\left(\frac{z}{\sqrt{2}}+\frac{1}{8}\pi\right)$;

(c) $N_0(z) = N_1(z) = \sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}$; (d) $\log_{10}\{\sqrt{z}N_0(z)\} = 0{\cdot}098-0{\cdot}307z$;

(e) $\phi_0(z) = (40{\cdot}51z+22{\cdot}5)^\circ$;

(f) $\phi_1(z) = (40{\cdot}51z+112{\cdot}5)^\circ = \phi_0(z)-90^\circ$.

49. (a) Verify that $K_0'(zi^{\frac{1}{2}}) = \operatorname{kei} z - i\ker_1 z$.

(b) If $E = x^p\{A_1 I_n(kx^q)+B_1 K_n(kx^q)\}$, find dE/dx.

$[x^{p-1}\{kqx^q[A_1 I_n'(kx^q)+B_1 K_n'(kx^q)]+p[A_1 I_n(kx^q)+B_1 K_n(kx^q)]\}.]$

50. Expand $J_1(re^{i\phi})$.

$$\left[\tfrac{1}{2}r\left\{\cos\phi-\frac{(\frac{1}{2}r)^2\cos 3\phi}{1!\,2!}+\frac{(\frac{1}{2}r)^4\cos 5\phi}{2!\,3!}-...+i\left[\sin\phi-\frac{(\frac{1}{2}r)^2\sin 3\phi}{1!\,2!}+...\right]\right\}.\right]$$

51. Show that, when $z \gg 1$, (a) $J_0(zi) \doteqdot \dfrac{e^z}{\sqrt{(2\pi z)}}$; (b) $J_1(zi) \doteqdot \dfrac{e^{z+\frac{1}{2}\pi i}}{\sqrt{(2\pi z)}}$.

52. Show that, when $z \gg 1$, (a) $Y_0(zi) \doteqdot iJ_0(zi)$; (b) $Y_1(zi) \doteqdot iJ_1(zi)$.

53. Verify that $J_n(z) = \dfrac{1}{2\pi i^n}\displaystyle\int_0^{2\pi} e^{iz\cos\theta}e^{in\theta}\,d\theta$.

54. Solve $\dfrac{d^2v}{dz^2}+az^2v = 0$. $\left[v = z^{\frac{1}{2}}\left\{A_1 J_{\frac{1}{4}}\left(\frac{a^{\frac{1}{2}}z^2}{2}\right)+B_1 J_{-\frac{1}{4}}\left(\frac{a^{\frac{1}{2}}z^2}{2}\right)\right\}.\right]$

55. Solve $\dfrac{d^2y}{dz^2}+a^4z^4y = 0$. $\left[y = z^{\frac{1}{2}}\left\{A_1 J_{\frac{1}{6}}\left(\frac{a^2z^3}{3}\right)+B_1 J_{-\frac{1}{6}}\left(\frac{a^2z^3}{3}\right)\right\}.\right]$

56. Solve $\dfrac{d^3y}{dz^3}+\dfrac{3}{z}\dfrac{d^2y}{dz^2}+\left(4-\dfrac{4n^2-1}{z^2}\right)\dfrac{dy}{dz}+\dfrac{4y}{z} = 0$.

$[y = A_1 J_n^2(z)+B_1 J_n(z)Y_n(z)+C_1 Y_n^2(z).]$

57. In a problem concerning the distribution of temperature θ in a solid circular cylinder, using cylindrical polar coordinates, the following equation occurs: $\dfrac{\partial^2\theta}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\theta}{\partial r}+\dfrac{\partial^2\theta}{\partial z^2} = 0$. Show that two particular solutions are $\theta = J_0(k_n r)\sinh(k_n z)$ and $\theta = J_0(k_n r)\cosh(k_n z)$.

58. If, in example 57, $\theta = 0$ when $r = a$ the radius of the cylinder, also if $\theta = 0$ when $z = 0$, the base of the cylinder being in the XY-plane, show that $k_1,..., k_n$ must be the roots of $J_0(k_n a) = 0$. Taking $\theta = \sum_{n=1}^{n} A_n J_0(k_n r)$ when $z = l$, at the top of the cylinder, show that

$$\theta = \sum_{n=1}^{n} A_n \frac{\sinh(k_n z)}{\sinh(k_n l)} J_0(k_n r).$$

59. Multiply $J_0''(z)+\frac{1}{z}J_0'(z)+J_0(z)=0$ by $z^2J_0'(z)\,dz$, integrate by parts between 0 and z, and show that

$$\int_0^z zJ_0^2(z)\,dz = \tfrac{1}{2}z^2[J_0^2(z)+J_1^2(z)].$$

60. If y_1 and y_2 are two linearly independent solutions of Bessel's differential equation, then $W(y_1,y_2)=y_1y_2'-y_2y_1'=A_1/z$, where A_1 is not zero. Show that when

$$(a)\ y_1=I_\nu(z),\quad y_2=I_{-\nu}(z),\quad A_1=-2\sin\nu\pi/\pi;$$
$$(b)\ y_1=I_\nu(z),\quad y_2=K_\nu(z),\quad A_1=-1.$$

61. In example 60 show that when

$$(a)\ y_1=J_\nu(z),\quad y_2=Y_\nu(z),\quad A_1=2/\pi;$$
$$(b)\ y_1=H_\nu^{(1)}(z),\quad y_2=H_\nu^{(2)}(z),\quad A_1=-4i/\pi.$$

[Since $W(y_1,y_2)=A_1/z$ for all values of z, it is convenient to find A_1 when $z\to 0$ and all the functions assume simple forms. The approximate values of the functions are used for y_1, y_2, y_1', and y_2'. See example 21.]

62. Using the formulae in example 61, show that

$$\frac{d}{dz}\left\{\log\frac{Y_\nu(z)}{J_\nu(z)}\right\}=\frac{2}{\pi zJ_\nu(z)Y_\nu(z)},$$

and hence that [93]

$$\int^z\frac{dz}{zJ_\nu(z)Y_\nu(z)}=\tfrac{1}{2}\pi\log\frac{Y_\nu(z)}{J_\nu(z)}.$$

63. Using the formulae in example 61, show that [93]

$$\int^z\frac{dz}{zY_\nu^2(z)}=-\tfrac{1}{2}\pi\frac{J_\nu(z)}{Y_\nu(z)}.$$

64. Show that $J_{\nu+1}(z)Y_\nu(z)-J_\nu(z)Y_{\nu+1}(z)=2/\pi z.$ [Use formulae in 61.]

65. Verify that $2(n+1)zJ_n'(z)+z^2J_{n+2}(z)-\{2n(n+1)-z^2\}J_n(z)=0.$

66. Verify that $z^2J_n''(z)+\{z^2-n(n-1)\}J_n(z)-zJ_{n+1}(z)=0.$

67. Verify that $z^2(n+1)J_n''(z)+n(z^2-n^2+1)J_n(z)+z^2J_{n+1}'(z)=0.$

68. Show that the maximum and minimum values of $z^{-\nu}J_\nu(z)$ occur when $z^{-\nu}J_{\nu+1}(z)=0$, i.e. they correspond to the roots of $J_{\nu+1}(z)=0.$

69. Show that the zeros of $J_\nu(z)$ correspond to the maximum and minimum values of $z^{\nu+1}J_{\nu+1}(z)=0.$

70. Show that Taylor's theorem can be expressed symbolically thus [22]:

$$\psi(x+h)=e^{hd/dx}\psi(x).$$

If $h=z\cos\theta$, show that

$$\psi(x+z\cos\theta)=\left\{I_0(zd/dx)+2\sum_{n=1}^{\infty}\cos n\theta\, I_n(zd/dx)\right\}\psi(x).$$

71. Show that, when $R(\nu)>-1$ and $x\to 0$,

$$(a)\ lzJ_\nu(kz)K_{\nu+1}(lz)\to(k/l)^\nu;\quad (b)\ kzJ_{\nu+1}(kz)K_\nu(lz)\to 0.$$

LIST OF FORMULAE

1. Differential equations†

1. $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+k^2y=0$ $\qquad y=A_1J_0(kz)+B_1Y_0(kz)$.

2. $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}+\left(k^2-\dfrac{\nu^2}{z^2}\right)y=0$ $\qquad y=A_1J_\nu(kz)+B_1Y_\nu(kz)$, always;
$y=A_1J_n(kz)+B_1Y_n(kz)$, n integral;
$y=A_1J_\nu(kz)+B_1J_{-\nu}(kz)$, ν non-integral.

3. $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-k^2y=0$ $\qquad y=A_1I_0(kz)+B_1K_0(kz)$.

4. $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-\left(k^2+\dfrac{\nu^2}{z^2}\right)y=0$ $\qquad y=A_1I_\nu(kz)+B_1K_\nu(kz)$, always;
$y=A_1I_n(kz)+B_1K_n(kz)$, n integral;
$y=A_1I_\nu(kz)+B_1I_{-\nu}(kz)$, ν non-integral.

5. $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-ik^2y=0$ $\qquad y=A_1I_0(kzi^{\frac{1}{2}})+B_1K_0(kzi^{\frac{1}{2}})$;
$y=A_1J_0(kzi^{\frac{3}{2}})+B_1K_0(kzi^{\frac{1}{2}})$.

6. $\dfrac{d^2y}{dz^2}+\dfrac{1}{z}\dfrac{dy}{dz}-\left(ik^2+\dfrac{\nu^2}{z^2}\right)y=0$ $\qquad y=A_1I_\nu(kzi^{\frac{1}{2}})+B_1K_\nu(kzi^{\frac{1}{2}})$, always;
$y=A_1I_n(kzi^{\frac{1}{2}})+B_1K_n(kzi^{\frac{1}{2}})$, n integral;
$y=A_1I_\nu(kzi^{\frac{1}{2}})+B_1I_{-\nu}(kzi^{\frac{1}{2}})$, ν non-integral.

7. $\dfrac{d^4y}{dz^4}+\dfrac{2}{z}\dfrac{d^3y}{dz^3}-\dfrac{1}{z^2}\dfrac{d^2y}{dz^2}+\dfrac{1}{z^3}\dfrac{dy}{dz}-k^4y=0$
$y=A_1J_0(kz)+B_1Y_0(kz)+C_1I_0(kz)+D_1K_0(kz)$.

2. Functions of the first kind

8. $$J_0(z)=1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\frac{z^6}{2^2.4^2.6^2}+\ldots$$
$$=1-\frac{(\frac{1}{2}z)^2}{(1!)^2}+\frac{(\frac{1}{2}z)^4}{(2!)^2}-\frac{(\frac{1}{2}z)^6}{(3!)^2}+\ldots=\sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{2r}}{(r!)^2}.$$

9. $J_0(z)\simeq\sqrt{\left(\dfrac{2}{\pi z}\right)}\cos(z-\frac{1}{4}\pi)$, when $|z|$ is large enough and $-\frac{1}{2}\pi\leqslant$ phase $z\leqslant\frac{1}{2}\pi$.‡ At $z=10$ the error is of the order 1 per cent., but steadily decreases with increase in z. See also § 11, Chap. IV.

10. $$J_0(z)=\frac{1}{2\pi}\int_0^{2\pi}e^{iz\sin\theta}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}e^{iz\cos\theta}\,d\theta=\frac{1}{\pi}\int_0^{\pi}e^{iz\cos\theta}\,d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\cos(z\cos\theta)\,d\theta=\frac{1}{\pi}\int_0^{\pi}\cos(z\cos\theta)\,d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\cos(z\sin\theta)\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}e^{-iz\sin\theta}\,d\theta.$$

† For additional equations see examples at the end of Chapter II.
‡ To obtain the function for a different range of θ see § 5, Chap. IV.

11. $\int\limits_0^z J_0(z)\,dz = 2\sum\limits_{n=0}^{\infty} J_{2n+1}(z) = zJ_0(z)+\frac{1}{2}\pi z\{J_1(z)\mathbf{H}_0(z)-J_0(z)\mathbf{H}_1(z)\}.$

12. $$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\left\{1-\frac{(\frac{1}{2}z)^2}{(\nu+1)}+\frac{(\frac{1}{2}z)^4}{2!(\nu+1)(\nu+2)}-\frac{(\frac{1}{2}z)^6}{3!(\nu+1)(\nu+2)(\nu+3)}+\dots\right\}$$
$$= \frac{z^\nu}{2^\nu\Gamma(\nu+1)}\left\{1-\frac{z^2}{2(2\nu+2)}+\frac{z^4}{2.4(2\nu+2)(2\nu+4)}-\dots\right\}$$
$$= \sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{\nu+2r}}{r!\Gamma(\nu+r+1)} = \sum_{r=0}^{\infty}(-1)^r\frac{z^{\nu+2r}}{2^{\nu+2r}r!\Gamma(\nu+r+1)}.$$

13. When $|z|$ is large enough and $-\frac{1}{2}\pi < \text{phase } z < \frac{1}{2}\pi$,†
$J_\nu(z) \doteq$
$$\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\left[1-\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2}+\frac{(4\nu^2-1^2)\dots(4\nu^2-7^2)}{4!(8z)^4}-\dots\right]\cos(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)-\right.$$
$$\left.-\left[\frac{4\nu^2-1^2}{1!8z}-\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{3!(8z)^3}+\dots\right]\sin(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)\right\}.$$

See also § 11, Chap. IV.

14. $J_{-n}(z) = (-1)^n J_n(z)$; $J_n(-z) = (-1)^n J_n(z)$; $J_{-n}(z) = J_n(-z)$.

15. $zJ_\nu'(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$.

16. $zJ_\nu'(z) = -\nu J_\nu(z) + zJ_{\nu-1}(z)$.

17. $2J_\nu'(z) = J_{\nu-1}(z) - J_{\nu+1}(z)$, by addition from 15 and 16.

18. $\frac{2\nu}{z}J_\nu(z) = J_{\nu+1}(z) + J_{\nu-1}(z)$, by subtraction from 15 and 16.

19. $J_0'(z) = -J_1(z)$, from 15 when $\nu = 0$.

20. $\int\limits^z z^{-\nu}J_{\nu+1}(z)\,dz = -z^{-\nu}J_\nu(z)$; or $\frac{d}{dz}\left\{z^{-\nu}J_\nu(z)\right\} = -z^{-\nu}J_{\nu+1}(z)$.

21. $\left(\frac{1}{z}\frac{d}{dz}\right)^r\{z^{-\nu}J_\nu(z)\} = (-1)^r z^{-\nu-r}J_{\nu+r}(z)$.

22. $\int\limits^z z^\nu J_{\nu-1}(z)\,dz = z^\nu J_\nu(z)$; or $\frac{d}{dz}\{z^\nu J_\nu(z)\} = z^\nu J_{\nu-1}(z)$.

23. $\left(\frac{1}{z}\frac{d}{dz}\right)^r\{z^\nu J_\nu(z)\} = z^{\nu-r}J_{\nu-r}(z)$.

24. $J_\nu'(kz) = \frac{d\{J_\nu(kz)\}}{d(kz)}$, so $\frac{d}{dz}\{J_\nu(kz)\} = kJ_\nu'(kz)$.

25. $J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int\limits_0^\pi e^{\pm iz\cos\theta}\sin^{2\nu}\theta\,d\theta = \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int\limits_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta.$
$R(\nu) > -\frac{1}{2}$.

26. $J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int\limits_0^{\frac{1}{2}\pi}\cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta.\quad R(\nu) > -\frac{1}{2}.$

27. $J_\nu(z) = \frac{(2\nu-1)(\frac{1}{2}z)^{\nu-1}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int\limits_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\sin^{2\nu-2}\theta\cos\theta\,d\theta.\quad R(\nu) > +\frac{1}{2}.$

† To obtain the function for a different range of θ see § 5, Chap. IV.

28. $J_\nu(z) = \dfrac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\displaystyle\int_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\cos^{2\nu}\theta\,d\theta.\quad R(\nu) > -\tfrac{1}{2}.$

29. $J_\nu(z) = \dfrac{2(\frac{1}{2}z)^{-\nu}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\nu)}\displaystyle\int_1^{\infty}\frac{\sin(zt)}{(t^2-1)^{\nu+\frac{1}{2}}}\,dt.\quad z > 0;\ \tfrac{1}{2} > R(\nu) > -\tfrac{1}{2}.$

30. $J_n(z) = \dfrac{i^{-n}}{2\pi}\displaystyle\int_0^{2\pi}e^{iz\cos\theta}e^{in\theta}\,d\theta = \frac{i^{-n}}{2\pi}\int_0^{2\pi}e^{iz\cos\theta}\cos n\theta\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta - z\sin\theta)\,d\theta.$

31. $\displaystyle\int_0^z J_\nu(z)\,dz = 2\sum_{r=0}^{\infty}J_{\nu+2r+1}(z);\quad \int_0^{\infty}J_\nu(z)\,dz = 1.\quad R(\nu) > -1.$

32. $J_{\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)}\sin z = Y_{-\frac{1}{2}}(z) = \mathbf{H}_{-\frac{1}{2}}(z).$

33. $J_{-\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)}\cos z = -Y_{\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)} - \mathbf{H}_{\frac{1}{2}}(z).$

34. $J_{\frac{1}{2}}(zi) = \sqrt{\left(\dfrac{2i}{\pi z}\right)}\sinh z = (1+i)\sqrt{\left(\dfrac{1}{\pi z}\right)}\sinh z.$

35. $J_{-\frac{1}{2}}(zi) = \sqrt{\left(\dfrac{2}{\pi z i}\right)}\cosh z = (1-i)\sqrt{\left(\dfrac{1}{\pi z}\right)}\cosh z.$

36. $$J_{n+\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\sin(z-\tfrac{1}{2}n\pi)\sum_{r=0}^{\leqslant\frac{1}{2}n}\frac{(-1)^r(n+2r)!}{(2r)!\,(n-2r)!\,(2z)^{2r}} + \cos(z-\tfrac{1}{2}n\pi)\sum_{r=0}^{\leqslant\frac{1}{2}(n-1)}\frac{(-1)^r(n+2r+1)}{(2r+1)!\,(n-2r-1)\,(2z)^{2r+1}}\right\}.$$

37. $J_{n+\frac{1}{2}}(z) = (-1)^n z^{n+\frac{1}{2}}\left(\dfrac{1}{z}\dfrac{d}{dz}\right)^n\{z^{-\frac{1}{2}}J_{\frac{1}{2}}(z)\}$, from 21′ when $\nu = \frac{1}{2}$ and $r = n$.

38. $$J_{-n-\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\cos(z+\tfrac{1}{2}n\pi)\sum_{r=0}^{\leqslant\frac{1}{2}n}\frac{(-1)^r(n+2r)!}{(2r)!\,(n-2r)!\,(2z)^{2r}} - \sin(z+\tfrac{1}{2}n\pi)\sum_{r=0}^{\leqslant\frac{1}{2}(n-1)}\frac{(-1)^r(n+2r+1)!}{(2r+1)!\,(n-2r-1)!\,(2z)^{2r+1}}\right\}.$$

39. $J_{-n-\frac{1}{2}}(z) = z^{n+\frac{1}{2}}\left(\dfrac{1}{z}\dfrac{d}{dz}\right)^n\{z^{-\frac{1}{2}}J_{-\frac{1}{2}}(z)\}$, from 23 when $\nu = -\frac{1}{2}$, $r = n$.

40. $$J_\mu(z)J_\nu(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(\mu+\nu+2r)_r(\frac{1}{2}z)^{\mu+\nu+2r}}{r!\,\Gamma(\mu+r+1)\Gamma(\nu+r+1)},$$

where $(\mu+\nu+2r)_r = (\mu+\nu+2r)\ldots(\mu+\nu+r+1).$

41. $$J_\mu(az)J_\nu(bz) = \frac{(\frac{1}{2}az)^\mu(\frac{1}{2}bz)^\nu}{\Gamma(\nu+1)}\sum_{r=0}^{\infty}(-1)^r\frac{F(-r,\,-\mu-r;\,\nu+1;\,b^2/a^2)(\frac{1}{2}az)^{2r}}{r!\,\Gamma(\mu+r+1)}.$$

42. $\displaystyle\int_0^{\frac{1}{2}\pi}J_\nu(z\sin\theta)\sin^{\nu+1}\theta\cos^{2\mu+1}\theta\,d\theta = \frac{2^\mu\Gamma(\mu+1)}{z^{\mu+1}}J_{\mu+\nu+1}(z).\quad R(\mu)$ and $R(\nu) > -1.$

43. $\displaystyle\int_0^\infty \frac{J_\mu(az)J_\nu(bz)}{z^\beta}\,dz = \frac{b^\nu\Gamma\{\frac12(\mu+\nu-\beta+1)\}}{2^\beta a^{\nu-\beta+1}\Gamma(\nu+1)\Gamma\{\frac12(\mu-\nu+\beta+1)\}}\times$

$\times F[\frac12(-\mu+\nu-\beta+1), \frac12(\mu+\nu-\beta+1); \nu+1; b^2/a^2].$

$R(\beta) > -1$, $R(\mu+\nu-\beta+1) > 0$, and $a > b > 0$.

44. $\displaystyle\int_0^\infty e^{-\alpha z}J_\nu(\beta z)\,dz = \frac{1}{\sqrt{(\alpha^2+\beta^2)}}\left\{\frac{\sqrt{(\alpha^2+\beta^2)}-\alpha}{\beta}\right\}^\nu.$ $R(\alpha\pm i\beta) > 0$; $R(\nu) > -1$.

45. $\displaystyle\int_0^\infty e^{-\alpha z}J_\nu(\beta z)\,z^\nu\,dz = \frac{(2\beta)^\nu\Gamma(\nu+\frac12)}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac12}}.$ $R(\alpha\pm i\beta) > 0$; $R(\nu) > -\frac12$.

46. $\displaystyle\int_0^\infty e^{-az}J_\nu(bz)z^{\mu-1}\,dz = \frac{(\frac12 b/a)^\nu\Gamma(\mu+\nu)}{a^\mu\Gamma(\nu+1)}F[\frac12(\mu+\nu), \frac12(\mu+\nu+1); \nu+1; -b^2/a^2].$

$R(\mu+\nu) > 0$; $R(a\pm ib) > 0$.

47. $\displaystyle\int_0^z J_\nu(kz)J_\nu(lz)z\,dz = \frac{z}{k^2-l^2}\{lJ_\nu(kz)J_\nu'(lz) - kJ_\nu(lz)J_\nu'(kz)\}$

$\displaystyle= \frac{z}{k^2-l^2}\{kJ_\nu(lz)J_{\nu+1}(kz) - lJ_\nu(kz)J_{\nu+1}(lz)\}$

$\displaystyle= \frac{z}{k^2-l^2}\{lJ_{\nu-1}(lz)J_\nu(kz) - kJ_{\nu-1}(kz)J_\nu(lz)\}.$

$R(\nu) > -1$; $k \neq l$.

48. $\displaystyle\int_0^z J_\nu^2(kz)z\,dz = \frac12 z^2\left\{J_\nu'^2(kz) + \left(1-\frac{\nu^2}{k^2z^2}\right)J_\nu^2(kz)\right\}.$ $R(\nu) > -1$.

See also the integrals for cylinder functions in § 5.

49. $e^{\frac12 z(t-t^{-1})} = \displaystyle\sum_{n=-\infty}^{\infty} t^n J_n(z).$ $t \neq 0$.

50. $e^{iz\sin\theta} = J_0(z) + 2\{J_2(z)\cos 2\theta + J_4(z)\cos 4\theta + \dots\} +$
$+ 2i\{J_1(z)\sin\theta + J_3(z)\sin 3\theta + \dots\}.$

51. $\cos(z\sin\theta) = J_0(z) + 2\{J_2(z)\cos 2\theta + J_4(z)\cos 4\theta + \dots\}.$

52. $\sin(z\sin\theta) = 2\{J_1(z)\sin\theta + J_3(z)\sin 3\theta + \dots\}.$

53. $e^{iz\cos\theta} = J_0(z) - 2\{J_2(z)\cos 2\theta - J_4(z)\cos 4\theta + \dots\} +$
$+ 2i\{J_1(z)\cos\theta - J_3(z)\cos 3\theta + \dots\}.$

54. $\cos(z\cos\theta) = J_0(z) - 2\{J_2(z)\cos 2\theta - J_4(z)\cos 4\theta + \dots\}.$

55. $\sin(z\cos\theta) = 2\{J_1(z)\cos\theta - J_3(z)\cos 3\theta + \dots\}.$

56. $\cos z = J_0(z) - 2\{J_2(z) - J_4(z) + J_6(z) - \dots\},$
from 51, $\theta = \frac12\pi$, or from 54, $\theta = 0$.

57. $\sin z = 2\{J_1(z) - J_3(z) + J_5(z) - \dots\}$, from 55, $\theta = 0$, or from 52, $\theta = \frac12\pi$.

3. Functions of the second kind

58. $\displaystyle Y_0(z) = \frac{2}{\pi}\{\gamma + \log(\tfrac12 z)\}J_0(z) - \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r(\frac12 z)^{2r}}{(r!)^2}\left\{1 + \frac12 + \frac13 + \dots + \frac1r\right\}.$

59. $\displaystyle Y_0(z) \doteq \sqrt{\left(\frac{2}{\pi z}\right)}\sin(z - \tfrac14\pi)$, when $|z|$ is large enough, $-\frac12\pi < \text{phase } z < \frac12\pi$.

See also § 11, Chap. IV.

60. $\int_0^z Y_0(z)\,dz = zY_0(z) + \frac{1}{2}\pi z\{Y_1(z)\mathbf{H}_0(z) - Y_0(z)\mathbf{H}_1(z)\}$.

61. $Y_n(z) = \frac{2}{\pi}\{\gamma + \log\frac{1}{2}z\}J_n(z) - \frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{2}{z}\right)^{n-2r} -$

$$\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}\left\{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{r}+1+\frac{1}{2}+...+\frac{1}{n+r}\right\}.$$

The term in the last series is $\left\{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right\}$, when $r = 0$.

62. $Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$. If $\nu = n$ an integer, $Y_n(z)$ is the limit of the fraction when $\nu \to n$.

63. When $|z|$ is large enough and $-\frac{1}{2}\pi \leqslant$ phase $z \leqslant \frac{1}{2}\pi$,

$$Y_\nu(z) \doteq \sqrt{\left(\frac{2}{\pi z}\right)}\left[\left\{1 - \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2} + \frac{(4\nu^2-1^2)...(4\nu^2-7^2)}{4!(8z)^4} - ...\right]\sin(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi) + \left[\frac{4\nu^2-1^2}{1!8z} - \frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{3!(8z)^3} + ...\right]\cos(z-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)\right\}.$$

See also § 11, Chap. IV.

64. $Y_\nu(z)$ satisfies recurrence formulae of the type given in 15 to 23 inclusive: $Y_n(z)$ satisfies the form given in 14.

65. $Y'_\nu(kz) = \frac{dY_\nu(kz)}{d(kz)}$, so $\frac{dY_\nu(kz)}{dz} = kY'_\nu(kz)$.

66. $Y_\nu(z) = -\frac{2(\frac{1}{2}z)^{-\nu}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\nu)}\int_1^\infty \frac{\cos(zt)\,dt}{(t^2-1)^{\nu+\frac{1}{2}}}$. $z > 0$; $\frac{1}{2} > R(\nu) > -\frac{1}{2}$.

67. $Y_{\frac{1}{2}}(z) = -J_{-\frac{1}{2}}(z) = -\sqrt{\left(\frac{2}{\pi z}\right)}\cos z$; $Y_{n+\frac{1}{2}}(z) = (-1)^{n+1}J_{-n-\frac{1}{2}}(z)$.

68. $Y_{-\frac{1}{2}}(z) = J_{\frac{1}{2}}(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\sin z$; $Y_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z)$.

69. $\int^z J_\nu(z)Y_\nu(z)z\,dz$; $\int^z Y_\nu^2(z)z\,dz$. Since $J_\nu(z)$ and $Y_\nu(z)$ are both cylinder functions, these integrals can be obtained from (78), (79). See Chap. VI and p. 101.

4. Functions of the third kind

70. $H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z)$; $H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$; $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ satisfy recurrence formulae of the form 14 to 23 inclusive: also 74 to 79.

71. When $|z|$ is large enough and $-\frac{1}{2}\pi \leqslant$ phase $z \leqslant \frac{1}{2}\pi$,

$$H_\nu^{(1)}(z) \doteq \sqrt{\left(\frac{2}{\pi z}\right)}e^{i(z-\frac{1}{4}\pi-\frac{1}{2}\nu\pi)}\left\{1 - \frac{4\nu^2-1^2}{1!(8zi)} + \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8zi)^2} - ...\right\}.$$

72. $H_\nu^{(2)}(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)} e^{-i(z-\frac{1}{4}\pi-\frac{1}{2}\nu\pi)} \left\{1+\frac{4\nu^2-1^2}{1!(8zi)}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8zi)^2}+\ldots\right\}.$

To obtain the functions for a different range of θ, see §§ 5 and 12, Chap IV: for other formulae see § 11, Chap. IV. For angle range see 71.

73. $H_{n+\frac{1}{2}}^{(1)}(z) = J_{n+\frac{1}{2}}(z)+i(-1)^{n+1}J_{-n-\frac{1}{2}}(z)$; $H_{n+\frac{1}{2}}^{(2)}(z) = J_{n+\frac{1}{2}}(z)+i(-1)^{n}J_{-n-\frac{1}{2}}(z)$.

5. Cylinder functions†

74. $z\mathfrak{C}_\nu'(z) = \nu\mathfrak{C}_\nu(z)-z\mathfrak{C}_{\nu+1}(z)$.

75. $z\mathfrak{C}_\nu'(z) = -\nu\mathfrak{C}_\nu(z)+z\mathfrak{C}_{\nu-1}(z)$.

76. $2\mathfrak{C}_\nu'(z) = \mathfrak{C}_{\nu-1}(z)-\mathfrak{C}_{\nu+1}(z)$.

77. $\frac{2\nu}{z}\mathfrak{C}_\nu(z) = \mathfrak{C}_{\nu-1}(z)+\mathfrak{C}_{\nu+1}(z)$.

78.
$$\begin{aligned}\int^z \mathfrak{C}_\nu(kz)\bar{\mathfrak{C}}_\nu(lz)z\,dz &= \frac{z}{k^2-l^2}\left\{\mathfrak{C}_\nu(kz)\frac{d}{dz}\bar{\mathfrak{C}}_\nu(lz)-\bar{\mathfrak{C}}_\nu(lz)\frac{d}{dz}\mathfrak{C}_\nu(kz)\right\}\\ &= \frac{z}{k^2-l^2}\{l\mathfrak{C}_\nu(kz)\bar{\mathfrak{C}}_\nu'(lz)-k\bar{\mathfrak{C}}_\nu(lz)\mathfrak{C}_\nu'(kz)\}\\ &= \frac{z}{k^2-l^2}\{k\bar{\mathfrak{C}}_\nu(lz)\mathfrak{C}_{\nu+1}(kz)-l\mathfrak{C}_\nu(kz)\bar{\mathfrak{C}}_{\nu+1}(lz)\}\\ &= \frac{z}{k^2-l^2}\{l\bar{\mathfrak{C}}_{\nu-1}(lz)\mathfrak{C}_\nu(kz)-k\mathfrak{C}_{\nu-1}(kz)\bar{\mathfrak{C}}_\nu(lz)\} \qquad (k\neq l).\end{aligned}$$

79. $\int^z \mathfrak{C}_\nu^2(kz)z\,dz = \frac{1}{2}z^2\{\mathfrak{C}_\nu^2(kz)-\mathfrak{C}_{\nu-1}(kz)\mathfrak{C}_{\nu+1}(kz)\} = \frac{1}{2}z^2\left\{\mathfrak{C}_\nu'^2(kz)+\left(1-\frac{\nu^2}{k^2z^2}\right)\mathfrak{C}_\nu^2(kz)\right\}.$

80. $\int^z \mathfrak{C}_\nu(kz)\bar{\mathfrak{C}}_\nu(kz)z\,dz = \frac{1}{4}z^2\{2\mathfrak{C}_\nu(kz)\bar{\mathfrak{C}}_\nu(kz)-\mathfrak{C}_{\nu-1}(kz)\bar{\mathfrak{C}}_{\nu+1}(kz)-\mathfrak{C}_{\nu+1}(kz)\bar{\mathfrak{C}}_{\nu-1}(kz)\}.$

In 78, 79, 80, when the lower limit is zero there is a restriction on ν. See Chap. VI.

6. Modified functions of the first kind

81.
$$\begin{aligned}I_0(z) = J_0(zi) &= 1+(\tfrac{1}{2}z)^2+\frac{(\frac{1}{2}z)^4}{(2!)^2}+\frac{(\frac{1}{2}z)^6}{(3!)^2}+\ldots\\ &= \sum_{r=0}^{\infty}\frac{(\frac{1}{2}z)^{2r}}{(r!)^2}.\end{aligned}$$

82.
$$\begin{aligned}I_0(z) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-z\cos\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi} e^{-z\cos\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-z\sin\theta}\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos(iz\cos\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cosh(z\cos\theta)\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\cosh(z\sin\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{z\sin\theta}\,d\theta.\end{aligned}$$

† A cylinder function satisfies 76 and 77. Since these can be derived from 74 and 75, it satisfies them also. $\mathbf{H}_\nu(z)$, $I_\nu(z)$, and $K_\nu(z)$ are not cylinder functions, since they do not satisfy 76 and 77.

83. $$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\left\{1+\frac{(\frac{1}{2}z)^2}{\nu+1}+\frac{(\frac{1}{2}z)^4}{2!(\nu+1)(\nu+2)}+\ldots\right\}$$
$$= \frac{z^\nu}{2^\nu\Gamma(\nu+1)}\left\{1+\frac{z^2}{2(2\nu+2)}+\frac{z^4}{2.4(2\nu+2)(2\nu+4)}+\ldots\right\}$$
$$= \sum_{r=0}^{\infty}\frac{(\frac{1}{2}z)^{\nu+2r}}{r!\Gamma(\nu+r+1)} = i^{-\nu}J_\nu(zi).$$

84. When $|z|$ is large enough and $0 \leqslant$ phase $z \leqslant \pi$,
$$I_\nu(z) \doteq \frac{e^z}{\sqrt{(2\pi z)}}\left\{1-\frac{4\nu^2-1^2}{1!8z}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2}-\ldots\right\}+$$
$$+e^{(\nu+\frac{1}{2})\pi i}\frac{e^{-z}}{\sqrt{(2\pi z)}}\left\{1+\frac{4\nu^2-1^2}{1!8z}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2}+\ldots\right\}.$$
When $R(z) > 0$ the second series can be neglected.

85. When $|z|$ is large enough and $-\pi <$ phase $z \leqslant 0$,
$$I_\nu(z) \doteq \frac{e^z}{\sqrt{(2\pi z)}}\left\{1-\frac{(4\nu^2-1^2)}{1!8z}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2}-\ldots\right\}+$$
$$+e^{-(\nu+\frac{1}{2})\pi i}\frac{e^{-z}}{\sqrt{(2\pi z)}}\left\{1+\frac{(4\nu^2-1^2)}{1!8z}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2}+\ldots\right\}.$$
When $R(z) > 0$ the second series can be neglected.

86. $I_{-n}(z) = I_n(z)$.

87. $I_\nu(z) = i^{-\nu}J_\nu(zi)$; $I_n(z) = i^nJ_n(-zi)$.

88. $zI_\nu'(z) = \nu I_\nu(z)+zI_{\nu+1}(z)$.

89. $zI_\nu'(z) = -\nu I_\nu(z)+zI_{\nu-1}(z)$.

90. $2I_\nu'(z) = I_{\nu-1}(z)+I_{\nu+1}(z)$, from 88 and 89 by addition.

91. $\frac{2\nu}{z}I_\nu(z) = I_{\nu-1}(z)-I_{\nu+1}(z)$, from 88 and 89 by subtraction.

92. $I_0'(z) = I_1(z)$.

93. $\int^z z^{-\nu}I_{\nu+1}(z)\,dz = z^{-\nu}I_\nu(z)$; or $\frac{d}{dz}\{z^{-\nu}I_\nu(z)\} = z^{-\nu}I_{\nu+1}(z)$.

94. $\left(\frac{1}{z}\frac{d}{dz}\right)^r\{z^{-\nu}I_\nu\} = z^{-\nu-r}I_{\nu+r}(z)$.

95. $\int^z z^\nu I_{\nu-1}(z)\,dz = z^\nu I_\nu(z)$; or $\frac{d}{dz}\{z^\nu I_\nu(z)\} = z^\nu I_{\nu-1}(z)$.

96. $\left(\frac{1}{z}\frac{d}{dz}\right)^r\{z^\nu I_\nu(z)\} = z^{\nu-r}I_{\nu-r}(z)$.

97. $I_\nu'(kz) = \frac{d}{d(kz)}\{I_\nu(kz)\}$, so $\frac{d}{dz}\{I_\nu(kz)\} = kI_\nu'(kz)$.

98. $I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta.$ $R(\nu) > -\frac{1}{2}$.

99. $I_\nu(z) = \dfrac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \displaystyle\int_0^\pi \cosh(z\cos\theta)\sin^{2\nu}\theta\, d\theta. \quad R(\nu) > -\tfrac{1}{2}.$

100. $I_\nu(z) = \dfrac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \displaystyle\int_0^{\frac{1}{2}\pi} \cosh\begin{Bmatrix} z\cos\theta \\ z\sin\theta \end{Bmatrix} \begin{matrix}\sin^{2\nu}\theta \\ \cos^{2\nu}\theta\end{matrix}\, d\theta. \quad R(\nu) > -\tfrac{1}{2}.$

101. $I_\nu(z) = \dfrac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \displaystyle\int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}}\cosh zt\, dt. \quad R(\nu) > -\tfrac{1}{2}.$

102. $I_n(z) = \dfrac{(-1)^n}{2\pi} \displaystyle\int_0^{2\pi} e^{-z\cos\theta}\cos n\theta\, d\theta = \dfrac{(-1)^n}{\pi} \displaystyle\int_0^{\pi} e^{-z\cos\theta}\cos n\theta\, d\theta.$

103. $I_n(z) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{z\cos\theta}\cos n\theta\, d\theta = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} e^{z\cos\theta}\cos n\theta\, d\theta.$

104. $I_{\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)}\sinh z = e^{-\frac{1}{4}\pi i}J_{+\frac{1}{2}}(zi) = i^{-\frac{1}{2}}J_{\frac{1}{2}}(zi).$

105. $I_{-\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)}\cosh z = e^{\frac{1}{4}\pi i}J_{-\frac{1}{2}}(zi) = i^{\frac{1}{2}}J_{-\frac{1}{2}}(zi).$

106. $$\int_0^z I_\nu(kz)I_\nu(lz)z\, dz = \frac{z}{k^2-l^2}\left\{I_\nu(lz)\frac{d}{dz}I_\nu(kz) - I_\nu(kz)\frac{d}{dz}I_\nu(lz)\right\}$$
$$= \frac{z}{k^2-l^2}\{kI_\nu(lz)I_\nu'(kz) - l\,I_\nu(kz)I_\nu'(lz)\}$$
$$= \frac{z}{k^2-l^2}\{kI_\nu(lz)I_{\nu+1}(kz) - l\,I_\nu(kz)I_{\nu+1}(lz)\}$$
$$= \frac{z}{k^2-l^2}\{kI_{\nu-1}(kz)I_\nu(lz) - l\,I_{\nu-1}(lz)I_\nu(kz)\}. \quad R(\nu) > -1.$$

107. $\displaystyle\int_0^z I_\nu^2(kz)z\, dz = -\tfrac{1}{2}z^2\left\{I_\nu'^2(kz) - \left(1+\frac{\nu^2}{k^2z^2}\right)I_\nu^2(kz)\right\}. \quad R(\nu) > -1.$

108. $$\int_0^z I_\nu(kz)J_\nu(lz)z\, dz = \frac{z}{k^2+l^2}\{k\,J_\nu(lz)I_\nu'(kz) - l\,I_\nu(kz)J_\nu'(lz)\},$$
$$= \frac{z}{k^2+l^2}\{k\,J_\nu(lz)I_{\nu+1}(kz) + l\,I_\nu(kz)J_{\nu+1}(lz)\}, \quad R(\nu) > -1.$$

109. By writing zi for z and $J_n(zi) = i^nI_n(z)$ in formulae 49 to 57 the corresponding expansions for the various functions can be obtained in terms of $I_n(z)$.

7. Modified functions of the second kind

110. $$K_0(z) = -\{\gamma+\log(\tfrac{1}{2}z)\}I_0(z) + \sum_{r=1}^{\infty}\frac{(\frac{1}{2}z)^{2r}}{(r!)^2}\left\{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{r}\right\}$$
$$= \tfrac{1}{2}\pi i\{I_0(z)+iY_0(zi)\}.$$

111. $K_0(z) \doteqdot \sqrt{\left(\dfrac{\pi}{2z}\right)}e^{-z}$, when $|z|$ is large enough and $-\pi < \text{phase } z \leqslant \pi$.

112. $$K_n(z) = \frac{1}{2}\sum_{r=0}^{n-1}\frac{(-1)^r(n-r-1)!}{r!}\left(\frac{2}{z}\right)^{n-2r} + (-1)^{n+1}\sum_{r=0}^{\infty}\frac{(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}\{\log\tfrac{1}{2}z-\tfrac{1}{2}[\psi(r+1)+\psi(n+r+1)]\},$$

where $$\psi(r+1) = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{r}\right)-\gamma, \quad \psi(1) = -\gamma,$$

$$\psi(n+r+1) = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+r}\right)-\gamma; \quad \gamma = 0{\cdot}5772\ldots.$$

113. $K_\nu(z) = \dfrac{\frac{1}{2}\pi\{I_{-\nu}(z)-I_\nu(z)\}}{\sin\nu\pi}$, and $K_n(z)$ is the limit when $\nu \to n$.

114. When $|z|$ is large enough and $-\pi < \text{phase } z \leqslant \pi$,

$$K_\nu(z) = \sqrt{\left(\frac{\pi}{2z}\right)}e^{-z}\left\{1+\frac{4\nu^2-1^2}{1!\,8z}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!\,(8z)^2}+\ldots+\frac{(4\nu^2-1^2)\ldots\{4\nu^2-(2r-3)^2\}}{(r-1)!\,(8z)^{r-1}}+\ldots\right\}.$$

115. $K_{-\nu}(z) = K_\nu(z) = \frac{1}{2}\pi i^{\nu+1}H_\nu^{(1)}(zi)$.

116. $zK'_\nu(z) = \nu K_\nu(z)-zK_{\nu+1}(z)$.

117. $zK'_\nu(z) = -\nu K_\nu(z)-zK_{\nu-1}(z)$.

118. $2K'_\nu(z) = -[K_{\nu-1}(z)+K_{\nu+1}(z)]$, from 116 and 117 by addition.

119. $\dfrac{2\nu}{z}K_\nu(z) = K_{\nu+1}(z)-K_{\nu-1}(z)$, from 116 and 117 by subtraction.

120. $K'_0(z) = -K_1(z)$.

121. $\displaystyle\int^z z^{-\nu}K_{\nu+1}(z)\,dz = -z^{-\nu}K_\nu(z)$; or $\dfrac{d}{dz}\{z^{-\nu}K_\nu(z)\} = -z^{-\nu}K_{\nu+1}(z)$.

122. $\left(\dfrac{1}{z}\dfrac{d}{dz}\right)^r\{z^{-\nu}K_\nu(z)\} = (-1)^r z^{-\nu-r}K_{\nu+r}(z)$.

123. $\displaystyle\int^z z^{\nu}K_{\nu-1}(z)\,dz = -z^{\nu}K_\nu(z)$; or $\dfrac{d}{dz}\{z^{\nu}K_\nu(kz)\} = -z^{\nu}K_{\nu-1}(z)$.

124. $\left(\dfrac{1}{z}\dfrac{d}{dz}\right)^r\{z^{\nu}K_\nu(z)\} = (-1)^r z^{\nu-r}K_{\nu-r}(z)$.

125. $K'_\nu(kz) = \dfrac{d}{d(kz)}\{K_\nu(kz)\}$, so $\dfrac{d}{dz}\{K_\nu(kz)\} = kK'_\nu(kz)$.

126. $$K_\nu(z) = \frac{\sqrt{\pi}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-z\cosh\theta}\sinh^{2\nu}\theta\,d\theta, \quad R(\nu) > -\tfrac{1}{2},\ -\tfrac{1}{2}\pi < \text{phase } z < \tfrac{1}{2}\pi,$$

$$= \int_0^\infty e^{-z\cosh\theta}\cosh\nu\theta\,d\theta, \quad R(z) > 0.$$

127. $$K_\nu(z) = \frac{\sqrt{\pi}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt. \qquad R(\nu) > -\tfrac{1}{2},\ -\tfrac{1}{2}\pi < \text{phase } z < \tfrac{1}{2}\pi.$$

128. $K_{\frac{1}{2}}(z) = \sqrt{\left(\dfrac{\pi}{2z}\right)}e^{-z} = K_{-\frac{1}{2}}(z) = \frac{1}{2}\pi\{I_{-\frac{1}{2}}(z)-I_{\frac{1}{2}}(z)\}$, from 113.

129. $K_{n+\frac{1}{2}}(z) = \sqrt{\left(\frac{\pi}{2z}\right)}\, e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{r!\,(n-r)!\,(2z)^r}.$

130. $$\int_z^\infty K_\nu(kz)K_\nu(lz)z\,dz = \frac{z}{k^2-l^2}\left\{K_\nu(kz)\frac{d}{dz}K_\nu(lz) - K_\nu(lz)\frac{d}{dz}K_\nu(kz)\right\}$$
$$= \frac{z}{k^2-l^2}\{l\,K_\nu(kz)K_\nu'(lz) - k\,K_\nu(lz)K_\nu'(kz)\}$$
$$= \frac{z}{k^2-l^2}\{k\,K_\nu(lz)K_{\nu+1}(kz) - l\,K_\nu(kz)K_{\nu+1}(lz)\}$$
$$= \frac{z}{k^2-l^2}\{k\,K_{\nu-1}(kz)K_\nu(lz) - l\,K_{\nu-1}(lz)K_\nu(kz)\}.$$
$R(k+l) > 0.$

The limits of integration should be observed.

131. $$\int_z^\infty zK_\nu^2(kz)\,dz = \tfrac{1}{2}z^2\left\{K_\nu'^2(kz) - \left(1+\frac{\nu^2}{k^2z^2}\right)K_\nu^2(kz)\right\}. \quad R(k) > 0.$$

132. $$\int_0^z K_\nu(kz)J_\nu(lz)z\,dz = \frac{z}{k^2+l^2}\{k\,J_\nu(lz)K_\nu'(kz) - l\,K_\nu(kz)J_\nu'(lz)\} + \frac{(l/k)^\nu}{k^2+l^2}$$
$$= \frac{z}{k^2+l^2}\{l\,K_\nu(kz)J_{\nu+1}(lz) - k\,J_\nu(lz)K_{\nu+1}(kz)\} + \frac{(l/k)^\nu}{k^2+l^2}.$$
$R(\nu) > -1,\ k^2 \neq -l^2.$

133. $$\int_0^z K_\nu(kz)I_\nu(lz)z\,dz = \frac{z}{k^2-l^2}\{k\,I_\nu(lz)K_\nu'(kz) - l\,K_\nu(kz)I_\nu'(lz)\} + \frac{(l/k)^\nu}{k^2-l^2}$$
$$= -\frac{z}{k^2-l^2}\{k\,I_\nu(lz)K_{\nu+1}(kz) + l\,K_\nu(kz)I_{\nu+1}(lz)\} + \frac{(l/k)^\nu}{k^2-l^2}$$
$$= -\frac{z}{k^2-l^2}\{k\,K_{\nu-1}(kz)I_\nu(lz) + l\,I_{\nu-1}(lz)K_\nu(kz)\} + \frac{(l/k)^\nu}{k^2-l^2}.$$
$R(\nu) > -1,\ k^2 \neq l^2.$

8. Struve's function

134. $$\mathbf{H}_0(z) = \frac{2}{\pi}\left\{\frac{z}{1^2} - \frac{z^3}{1^2.3^2} + \frac{z^5}{1^2.3^2.5^2} - \ldots\right\} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin\frac{(z\sin\theta)}{(z\cos\theta)}\,d\theta.$$

135. $$\mathbf{H}_1(z) = \frac{2}{\pi}\left\{\frac{z^2}{1^2.3} - \frac{z^4}{1^2.3^2.5} + \frac{z^6}{1^2.3^2.5^2.7} - \ldots\right\}.$$

136. $$\mathbf{H}_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\sin^{2\nu}\theta\,d\theta = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}(1-t^2)^{\nu-\frac{1}{2}}\sin(zt)\,dt,$$
$R(\nu) > -\tfrac{1}{2},$
$$= \sum_{r=0}^\infty (-1)^r \frac{(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})}, \text{ always.}$$

136 a. When $|z|$ is large enough and $R(n+\frac{1}{2}-\nu) \geqslant 0$
$$\mathbf{H}_\nu(z) \doteqdot Y_\nu(z) + \frac{1}{\pi}\sum_{r=0}^{n-1}\frac{\Gamma(r+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-r)}\left(\frac{2}{z}\right)^{1+2r-\nu} + S_n.$$
where the remainder S_n can be made sufficiently small by taking $|z|$ large enough. The asymptotic expansion is used for $Y_\nu(z)$.

137. $z\mathbf{H}'_\nu(z) = \nu\mathbf{H}_\nu(z) - z\mathbf{H}_{\nu+1}(z) + \dfrac{2(\frac{1}{2}z)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$.

138. $z\mathbf{H}'_\nu(z) = -\nu\mathbf{H}_\nu(z) + z\mathbf{H}_{\nu-1}(z)$.

139. $2\mathbf{H}'_\nu(z) = \mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) + \dfrac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$, from 137 and 138 by addition.

140. $\dfrac{2\nu}{z}\mathbf{H}_\nu(z) = \mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) - \dfrac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$, from 137 and 138 by subtraction.

140 a. $\mathbf{H}'_0(z) = \dfrac{2}{\pi} - \mathbf{H}_1(z)$; $\displaystyle\int^z z\mathbf{H}_0(z)\,dz = z\mathbf{H}_1(z)$; or $\dfrac{d}{dz}\{z\mathbf{H}_1(z)\} = z\mathbf{H}_0(z)$.

141. $\displaystyle\int^z z^\nu\mathbf{H}_{\nu-1}(z)\,dz = z^\nu\mathbf{H}_\nu(z)$; or $\dfrac{d}{dz}\{z^\nu\mathbf{H}_\nu(z)\} = z^\nu\mathbf{H}_{\nu-1}(z)$.

142. $-\displaystyle\int^z z^{-\nu}\mathbf{H}_{\nu+1}(z)\,dz = z^{-\nu}\mathbf{H}_\nu(z) - \frac{z}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$;

or $\qquad \dfrac{d}{dz}\{z^{-\nu}\mathbf{H}_\nu(z)\} = \dfrac{1}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})} - z^{-\nu}\mathbf{H}_{\nu+1}(z)$.

143. $$J_\nu(z) + i\mathbf{H}_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} e^{iz\cos\theta}\sin^{2\nu}\theta\,d\theta$$
$$= \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} e^{iz\sin\theta}\cos^{2\nu}\theta\,d\theta. \quad R(\nu) > -\tfrac{1}{2}.$$

144. $\mathbf{H}_{\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)}(1-\cos z) = \sqrt{\left(\dfrac{2}{\pi z}\right)} - J_{-\frac{1}{2}}(z)$.

145. $\mathbf{H}_{-\frac{1}{2}}(z) = \sqrt{\left(\dfrac{2}{\pi z}\right)}\sin z = J_{\frac{1}{2}}(z)$.

146. $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-iz\cos\theta}\cos\theta\,d\theta = 2 - \pi\{\mathbf{H}_1(z) + i\,J_1(z)\}$.

147. $\displaystyle\int_0^{\frac{1}{2}\pi} e^{iz\cos\theta}\sin^2\theta\,d\theta = \frac{\pi}{2z}\{J_1(z) - i\mathbf{H}_1(z)\}$.

148. $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-iz\cos\theta}\cos^2\theta\,d\theta = \pi\left[\left\{J_0(z) - \frac{J_1(z)}{z}\right\} - i\left\{\mathbf{H}_0(z) - \frac{\mathbf{H}_1(z)}{z}\right\}\right]$.

149. $\displaystyle\int_0^z \mathbf{H}_0(z)\,dz = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left\{\frac{1-\cos(z\sin\theta)}{\sin\theta}\right\}d\theta$.

150. $\mathbf{H}_0(z) = \dfrac{4}{\pi}\displaystyle\sum_{n=0}^{\infty}\frac{J_{2n+1}(z)}{2n+1}$.

150 a. $\mathbf{H}''_\nu(z) + \dfrac{1}{z}\mathbf{H}'_\nu(z) + \left(1 - \dfrac{\nu^2}{z^2}\right)\mathbf{H}_\nu(z) - \dfrac{(\frac{1}{2}z)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} = 0.$

9. Hypergeometric and gamma functions

151. $F(\alpha,\beta,\gamma,z) = 1+\frac{\alpha\beta}{1!\gamma}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}z^2+\dots$. Series is absolutely convergent if $|z| < 1$. When $|z| = 1$, it is absolutely convergent if $R(\gamma-\alpha-\beta) > 0$.

152. $\int\limits_0^a F\left(\alpha,\beta,n,\frac{z^2}{a^2}\right)z^{2n-1}\,dz = \frac{a^{2n}}{2n}F(\alpha,\beta,n+1,1). \qquad R(n+1-\alpha-\beta) > 0.$

153. $\frac{\partial}{\partial z}F(\alpha,\beta,\gamma,\chi) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,\chi)\frac{\partial\chi}{\partial z}.$

154. $F(\alpha,\beta,\gamma,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \qquad R(\gamma) > 0, \quad R(\gamma-\alpha-\beta) > 0.$

155. $\Gamma(1+z) = z\Gamma(z)$; $\Gamma(n+1) = n!$ when n is a positive integer: $0! = 1$.

156. $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$; $\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})/\sqrt{\pi}$.

$$\Gamma(z) = \int\limits_0^\infty e^{-t}t^{z-1}\,dz, \quad R(z) > 0.$$

157. $1/\Gamma(1-z) = 0$, when z is a positive non-zero integer.

158. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$; $\Gamma(\frac{5}{2}) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$; $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$; $\Gamma(\frac{1}{3}) \doteqdot 2{\cdot}679$; $\Gamma(\frac{2}{3}) \doteqdot 1{\cdot}354$.

10. Ber and bei functions

159. $J_0(zi^{\frac{3}{2}}) = I_0(zi^{\frac{1}{2}}) = \operatorname{ber} z+i\operatorname{bei} z$. See Fig. 18 for $i^{\frac{1}{2}}$ and $i^{\frac{3}{2}}$.

160. $J_0(zi^{-\frac{3}{2}}) = I_0(zi^{-\frac{1}{2}}) = \operatorname{ber} z-i\operatorname{bei} z$.

161. $\operatorname{ber} z = \left\{1-\frac{z^4}{2^2.4^2}+\frac{z^8}{2^2.4^2.6^2.8^2}-\dots\right\} = \left\{1-\frac{(\frac{1}{2}z)^4}{(2!)^2}+\frac{(\frac{1}{2}z)^8}{(4!)^2}-\dots\right\}.$

162. $\operatorname{bei} z = \left\{\frac{z^2}{2^2}-\frac{z^6}{2^2.4^2.6^2}+\frac{z^{10}}{2^2.4^2.6^2.8^2.10^2}-\dots\right\} = \left\{(\frac{1}{2}z)^2-\frac{(\frac{1}{2}z)^6}{(3!)^2}+\frac{(\frac{1}{2}z)^{10}}{(5!)^2}-\dots\right\}.$

163. $J_\nu(zi^{\frac{3}{2}}) = i^\nu I_\nu(zi^{\frac{1}{2}}) = \operatorname{ber}_\nu z+i\operatorname{bei}_\nu z$.

164. $J_\nu(zi^{-\frac{3}{2}}) = i^{-\nu}I_\nu(zi^{-\frac{1}{2}}) = \operatorname{ber}_\nu z-i\operatorname{bei}_\nu z$.

165. $\frac{d}{d(kz)}J_0(kzi^{\frac{3}{2}}) = \frac{d}{d(kz)}(\operatorname{ber} kz+i\operatorname{bei} kz) = \operatorname{ber}' kz+i\operatorname{bei}' kz.$

166. $\frac{d}{dz}(\operatorname{ber} kz+i\operatorname{bei} kz) = k(\operatorname{ber}' kz+i\operatorname{bei}' kz).$

167. $J_0'(kzi^{\frac{3}{2}}) = i^{-\frac{3}{2}}(\operatorname{ber}' kz+i\operatorname{bei}' kz)$; $\operatorname{ber}' kz+i\operatorname{bei}' kz = i^{\frac{3}{2}}J_0'(kzi^{\frac{3}{2}})$.

168. Formulae for order ν are identical in form with 165–167 excepting that the order is inserted, e.g. $\operatorname{ber}_\nu z$.

169. $J_\nu(zi^{\frac{3}{2}}) = \operatorname{ber}_\nu z+i\operatorname{bei}_\nu z = M_\nu(z)\{\cos\theta_\nu(z)+i\sin\theta_\nu(z)\} = M_\nu(z)e^{i\theta_\nu(z)}$.

170. $\operatorname{ber}_\nu z = M_\nu(z)\cos\theta_\nu(z)$; $\operatorname{bei}_\nu z = M_\nu(z)\sin\theta_\nu(z)$.

171. $\theta_\nu(z) = \tan^{-1}(\operatorname{bei}_\nu z/\operatorname{ber}_\nu z)$; $-\theta_\nu(z) = \tan^{-1}(-\operatorname{bei}_\nu z/\operatorname{ber}_\nu z)$.

172. $J_\nu(zi^{-\frac{3}{2}}) = M_\nu(z)e^{-i\theta_\nu(z)}$.

173. $\operatorname{ber} z = M_0(z)\cos\theta_0(z)$.

174. $\operatorname{bei} z = M_0(z)\sin\theta_0(z)$.

175. $\operatorname{ber}'z = M_1(z)\cos(\theta_1-\frac{1}{4}\pi)$.

176. $\operatorname{bei}'z = M_1(z)\sin(\theta_1-\frac{1}{4}\pi)$.

177. $\operatorname{ber}'_\nu z = \frac{1}{2}\{M_{\nu+1}(z)\cos(\theta_{\nu+1}-\frac{1}{4}\pi)-M_{\nu-1}(z)\cos(\theta_{\nu-1}-\frac{1}{4}\pi)\}$.

178. $\operatorname{bei}'_\nu z = \frac{1}{2}\{M_{\nu+1}(z)\sin(\theta_{\nu+1}-\frac{1}{4}\pi)-M_{\nu-1}(z)\sin(\theta_{\nu-1}-\frac{1}{4}\pi)\}$.

179. $M_{-n}(z) = M_{+n}(z)$.

180. $\theta_{-n}(z) = \theta_n(z)+n\pi$.

181. $$\operatorname{ber}_\nu z = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)}\cos\tfrac{3}{4}(\nu+2r)\pi.$$

182. $$\operatorname{bei}_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)}\sin\tfrac{3}{4}(\nu+2r)\pi.$$

183. $$\operatorname{ber}'_\nu z = \sum_{r=0}^{\infty} \frac{(-1)^r\frac{1}{2}(\nu+2r)(\frac{1}{2}z)^{\nu+2r-1}}{r!\,\Gamma(\nu+r+1)}\cos\tfrac{3}{4}(\nu+2r)\pi.$$

184. $$\operatorname{bei}'_\nu z = \sum_{r=0}^{\infty} \frac{(-1)^r\frac{1}{2}(\nu+2r)(\frac{1}{2}z)^{\nu+2r-1}}{r!\,\Gamma(\nu+r+1)}\sin\tfrac{3}{4}(\nu+2r)\pi.$$

185.† When z is large enough and $-\frac{3}{4}\pi <$ phase $z < \frac{1}{4}\pi$,

$$\operatorname{ber}_\nu z \doteqdot \frac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\left\{\lambda_\nu(z)\cos\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)-\chi_\nu(z)\sin\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\}.$$

186.† $$\operatorname{bei}_\nu z \doteqdot \frac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\left\{\chi_\nu(z)\cos\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)+\lambda_\nu(z)\sin\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\},$$

where $\lambda_\nu(z) \doteqdot$

$$1-\frac{(4\nu^2-1^2)}{1!\,8z}\cos\tfrac{1}{4}\pi+\ldots+\frac{(-1)^r(4\nu^2-1^2)(4\nu^2-3^2)\ldots\{4\nu^2-(2r-1)^2\}}{r!(8z)^r}\cos\tfrac{1}{4}r\pi,$$

the number of the term being $r+1$; and

$\chi_\nu(z) \doteqdot$

$$\frac{(4\nu^2-1^2)}{1!\,8z}\sin\tfrac{1}{4}\pi-\ldots+\frac{(-1)^{r+1}(4\nu^2-1^2)(4\nu^2-3^2)\ldots\{4\nu^2-(2r-1)^2\}}{r!(8z)^r}\sin\tfrac{1}{4}r\pi,$$

the number of the term being r.

187.† $$\operatorname{ber}'_\nu z \doteqdot \frac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\left\{\psi_\nu(z)\cos\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)-\Omega_\nu(z)\sin\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\}.$$

188.† $$\operatorname{bei}'_\nu z \doteqdot \frac{e^{z/\sqrt{2}}}{\sqrt{(2\pi z)}}\left\{\Omega_\nu(z)\cos\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)+\psi_\nu(z)\sin\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\},$$

where $$\psi_\nu(z) = \tfrac{1}{2}\{\lambda_{\nu+1}(z)+\lambda_{\nu-1}(z)\}$$
$$= 1-\frac{(4\nu^2+1.3)}{1!\,8z}\cos\tfrac{1}{4}\pi+\ldots+$$
$$+\frac{(-1)^r(4\nu^2-1^2)(4\nu^2-3^2)\ldots\{4\nu^2-(2r-3^2)\}\{4\nu^2+(2r-1)(2r+1)\}}{r!(8z)^r}\cos\tfrac{1}{4}r\pi,$$

† 185, 186, 187, 188 are asymptotic expansions and the restriction concerning phase z applies in all cases.

the number of the term being $(r+1)$, and

$$\Omega_\nu(z) = \tfrac{1}{2}\{\chi_{\nu+1}(z)+\chi_{\nu-1}(z)\}$$
$$= \frac{(4\nu^2+1.3)}{1!\,8z}\sin\tfrac{1}{4}\pi-\ldots+$$
$$+\frac{(-1)^{r+1}(4\nu^2-1^2)(4\nu^2-3^2)\ldots\{4\nu^2-(2r-3)^2\}\{4\nu^2+(2r-1)(2r+1)\}}{r!(8z)^r}\sin\tfrac{1}{4}r\pi,$$

the number of the term being r.

189. $\displaystyle\int^z z\operatorname{ber}kz\,dz = \frac{z}{k}\operatorname{bei}'kz;$ or $\displaystyle\frac{d}{dz}(z\operatorname{bei}'kz) = kz\operatorname{ber}kz.$

190. $\displaystyle\int^z z\operatorname{bei}kz\,dz = -\frac{z}{k}\operatorname{ber}'kz;$ or $\displaystyle\frac{d}{dz}(z\operatorname{ber}'kz) = -kz\operatorname{bei}kz.$

191. $\displaystyle\int^z (\operatorname{ber}_\nu^2 kz+\operatorname{bei}_\nu^2 kz)z\,dz = \frac{z}{k}(\operatorname{ber}_\nu kz\operatorname{bei}_\nu' kz-\operatorname{bei}_\nu kz\operatorname{ber}_\nu' kz).$

192. $\displaystyle\int^z (\operatorname{ber}^2kz+\operatorname{bei}^2kz)z\,dz = \frac{z}{k}M_0(kz)M_1(kz)\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi).$

193. $\displaystyle\int^z (\operatorname{ber}_\nu'^2 kz+\operatorname{bei}_\nu'^2 kz)z\,dz = \frac{z}{k}(\operatorname{ber}_\nu kz\operatorname{ber}_\nu' kz+\operatorname{bei}_\nu kz\operatorname{bei}_\nu' kz).$

194. $\displaystyle\int^z (\operatorname{ber}'^2kz+\operatorname{bei}'^2kz)z\,dz = \frac{z}{k}M_0(kz)M_1(kz)\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi).$

195. $\displaystyle\int^z z\operatorname{ber}kz\operatorname{bei}kz\,dz = \tfrac{1}{2}z^2\{\operatorname{ber}kz\operatorname{bei}kz+\operatorname{ber}_1 kz\operatorname{bei}_1 kz\}$
$$= \tfrac{1}{4}z^2\{2\operatorname{ber}kz\operatorname{bei}kz+(\operatorname{ber}'^2kz-\operatorname{bei}'^2kz)\}.$$

196. $\displaystyle\int^z z(\operatorname{ber}^2kz-\operatorname{bei}^2kz)\,dz = \tfrac{1}{2}z^2\{(\operatorname{ber}_1^2 kz-\operatorname{bei}_1^2 kz)+(\operatorname{ber}^2kz-\operatorname{bei}^2kz)\}$
$$= \tfrac{1}{2}z^2\{(\operatorname{ber}^2kz-\operatorname{bei}^2kz)-2\operatorname{ber}'kz\operatorname{bei}'kz\}.$$

197. $\operatorname{ber}_\nu^2 z+\operatorname{bei}_\nu^2 z = M_\nu^2(z).$

198. $\operatorname{ber}'^2z+\operatorname{bei}'^2z = M_1^2(z).$

199. $\operatorname{ber}z\operatorname{bei}'z-\operatorname{bei}z\operatorname{ber}'z = M_0(z)M_1(z)\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi).$

200. $\operatorname{ber}z\operatorname{ber}'z+\operatorname{bei}z\operatorname{bei}'z = M_0(z)M_1(z)\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi).$

11. Ker and kei functions

201. $K_0(zi^{\frac{1}{2}}) = \ker z+i\operatorname{kei}z = \tfrac{1}{2}\pi i H_0^{(1)}(zi^{\frac{3}{2}}).$

202. $K_0(zi^{-\frac{1}{2}}) = \ker z-i\operatorname{kei}z = \tfrac{1}{2}\pi i H_0^{(1)}(zi^{\frac{1}{2}}).$

203. $\ker z = (\log 2-\gamma-\log z)\operatorname{ber}z+\tfrac{1}{4}\pi\operatorname{bei}z-$

$$-\frac{(\frac{1}{2}z)^4}{(2!)^2}(1+\tfrac{1}{2})+\frac{(\frac{1}{2}z)^8}{(4!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})-\ldots.\quad [\gamma = 0{\cdot}5772\ldots, \log 2-\gamma = 0{\cdot}1159\ldots.]$$

204. $\displaystyle\operatorname{kei}z = (\log 2-\gamma-\log z)\operatorname{bei}z-\tfrac{1}{4}\pi\operatorname{ber}z+(\tfrac{1}{2}z)^2-\frac{(\frac{1}{2}z)^6}{(3!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3})+\ldots.$

205. $i^{-\nu}K_\nu(zi^{\frac{1}{2}}) = \ker_\nu z+i\operatorname{kei}_\nu z.$

206. $K_\nu(zi^{\frac{1}{2}}) = i^\nu(\ker_\nu z+i\operatorname{kei}_\nu z) = \tfrac{1}{2}\pi i^{\nu+1}H_\nu^{(1)}(zi^{\frac{3}{2}}).$

207. $K_\nu(zi^{-\frac{1}{2}}) = i^{-\nu}(\ker_\nu z-i\operatorname{kei}_\nu z) = \tfrac{1}{2}\pi i^{\nu+1}H_\nu^{(1)}(zi^{\frac{1}{2}}).$

208. $\dfrac{d}{d(kz)}K_\nu(kzi^{\frac{1}{2}}) = i^\nu(\mathrm{ker}'_\nu kz + i\,\mathrm{kei}'_\nu kz).$

209. $\dfrac{d}{dz}(\mathrm{ker}_\nu kz + i\,\mathrm{kei}_\nu kz) = k(\mathrm{ker}'_\nu kz + i\,\mathrm{kei}'_\nu kz).$

210. $K'_\nu(kzi^{\frac{1}{2}}) = i^{\nu-\frac{1}{2}}(\mathrm{ker}'_\nu kz + i\,\mathrm{kei}'_\nu kz).$

211. $K_\nu(zi^{\frac{1}{2}}) = N_\nu(z)e^{i\{\phi_\nu(z)+\frac{1}{2}\nu\pi\}} = N_\nu(z)\{\cos(\phi_\nu(z)+\frac{1}{2}\nu\pi)+i\sin(\phi_\nu(z)+\frac{1}{2}\nu\pi)\}.$

212. $N_\nu(z)e^{i\phi_\nu(z)} = \mathrm{ker}_\nu z + i\,\mathrm{kei}_\nu z.$

213. $N_\nu(z)e^{i\phi_\nu(z)} = N_\nu(z)\{\cos\phi_\nu(z)+i\sin\phi_\nu(z)\}.$

214. $\mathrm{ker}_\nu z = N_\nu(z)\cos\phi_\nu(z).\quad (2/\pi)N_\nu(z)e^{-i\{\phi_\nu(z)+(\nu+\frac{1}{2})\pi\}} = H_\nu^{(1)}(zi^{\frac{1}{2}}).$

215. $\mathrm{kei}_\nu z = N_\nu(z)\sin\phi_\nu(z).$

216. $\phi_\nu(z) = \tan^{-1}\left(\dfrac{\mathrm{kei}_\nu z}{\mathrm{ker}_\nu z}\right);\quad -\phi_\nu(z) = \tan^{-1}\left(-\dfrac{\mathrm{kei}_\nu z}{\mathrm{ker}_\nu z}\right).$

217. $K_\nu(zi^{-\frac{1}{2}}) = N_\nu(z)e^{-i\{\phi_\nu(z)+\frac{1}{2}\nu\pi\}}.$

218. $\mathrm{ker}\, z = N_0(z)\cos\phi_0(z).$

219. $\mathrm{kei}\, z = N_0(z)\sin\phi_0(z).$

220. $\mathrm{ker}' z = N_1(z)\cos(\phi_1-\frac{1}{4}\pi).$

221. $\mathrm{kei}' z = N_1(z)\sin(\phi_1-\frac{1}{4}\pi).$

222. $\mathrm{ker}'_\nu z = \frac{1}{2}\{N_{\nu+1}(z)\cos(\phi_{\nu+1}-\frac{1}{4}\pi)-N_{\nu-1}(z)\cos(\phi_{\nu-1}-\frac{1}{4}\pi)\}.$

223. $\mathrm{kei}'_\nu z = \frac{1}{2}\{N_{\nu+1}(z)\sin(\phi_{\nu+1}-\frac{1}{4}\pi)-N_{\nu-1}(z)\sin(\phi_{\nu-1}-\frac{1}{4}\pi)\}.$

224. $N_{-n}(z) = N_n(z).$

225. $\phi_{-n}(z) = \phi_n(z)+n\pi.$

226. $\mathrm{ker}_n z = (\log 2-\gamma-\log z)\mathrm{ber}_n z + \frac{1}{4}\pi\,\mathrm{bei}_n z +$

$$+\frac{1}{2}\sum_{r=0}^{n-1}(-1)^{n+r}\frac{(n-r-1)!}{r!}\left(\frac{2}{z}\right)^{n-2r}\cos\{\tfrac{1}{4}(n+2r)\pi\}+$$

$$+\frac{1}{2}\sum_{r=0}^{\infty}(-1)^{n+r}\frac{(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}\left(1+\frac{1}{2}+\ldots+\frac{1}{r}+1+\frac{1}{2}+\ldots+\frac{1}{n+r}\right)\cos\{\tfrac{1}{4}(n+2r)\pi\}.$$

227. $\mathrm{kei}_n z = (\log 2-\gamma-\log z)\mathrm{bei}_n z - \frac{1}{4}\pi\,\mathrm{ber}_n z +$

$$+\frac{1}{2}\sum_{r=0}^{n-1}(-1)^{n+r}\frac{(n-r-1)!}{r!}\left(\frac{2}{z}\right)^{n-2r}\sin\{\tfrac{1}{4}(n+2r)\pi\}-$$

$$-\frac{1}{2}\sum_{r=0}^{\infty}(-1)^{n+r}\frac{(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}\left(1+\frac{1}{2}+\ldots+\frac{1}{r}+1+\frac{1}{2}+\ldots+\frac{1}{n+r}\right)\sin\{\tfrac{1}{4}(n+2r)\pi\}.$$

Formulae 226 and 227 can be obtained from 112 by taking

$$i^{-n}K_n(zi^{\frac{1}{2}}) = \mathrm{ker}_n z + i\,\mathrm{kei}_n z$$

and using (*a*), (*b*) example 54, Chap. VIII.

By differentiating 226 and 227, we get 228 and 229.

228. $\mathrm{ker}'_n z = (\log 2-\gamma-\log z)\mathrm{ber}'_n z - \dfrac{\mathrm{ber}_n z}{z} + \frac{1}{4}\pi\,\mathrm{bei}'_n z +$

$$+\frac{1}{4}\sum_{r=0}^{n-1}(-1)^{n+r}\frac{(2r-n)(n-r-1)!}{r!}\left(\frac{2}{z}\right)^{n+1-2r}\cos\{\tfrac{1}{4}(n+2r)\pi\}+$$

$$+\frac{1}{4}\sum_{r=0}^{\infty}(-1)^{n+r}\frac{(n+2r)(\frac{1}{2}z)^{n+2r-1}}{r!\,(n+r)!}\left(1+\frac{1}{2}+\ldots+\frac{1}{r}+1+\frac{1}{2}+\ldots+\frac{1}{n+r}\right)\cos\{(\tfrac{1}{4}n+2r)\pi\}.$$

229. $\operatorname{kei}_n' z = (\log 2 - \gamma - \log x)\operatorname{bei}_n' z - \dfrac{\operatorname{bei}_n z}{z} - \tfrac{1}{4}\pi \operatorname{ber}_n' z +$

$$+\frac{1}{4}\sum_{r=0}^{n-1}(-1)^{n+r}\frac{(2r-n)(n-r-1)!}{r!}\left(\frac{2}{z}\right)^{n+1-2r}\sin\{\tfrac{1}{4}(n+2r)\pi\} -$$

$$-\frac{1}{4}\sum_{r=0}^{\infty}(-1)^{n+r}\frac{(n+2r)(\frac{1}{2}z)^{n+2r-1}}{r!(n+r)!}\left(1+\frac{1}{2}+\dots\frac{1}{r}+1+\frac{1}{2}+\dots\frac{1}{n+r}\right)\sin\{\tfrac{1}{4}(n+2r)\pi\}.$$

230. When $|z|$ is large enough and $-\frac{7}{4}\pi <$ phase $z < \frac{5}{4}\pi$,

$$\operatorname{ker}_\nu z \doteqdot \sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{\lambda_\nu(-z)\cos\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)+\chi_\nu(-z)\sin\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\}.$$

231. $\operatorname{kei}_\nu z$

$$\doteqdot \sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{\chi_\nu(-z)\cos\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)-\lambda_\nu(-z)\sin\left(\frac{z}{\sqrt{2}}+\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\}.$$

232. $\operatorname{ker}_\nu' z$

$$\doteqdot -\sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{\psi_\nu(-z)\cos\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)+\Omega_\nu(-z)\sin\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\}.$$

233. $\operatorname{kei}_\nu' z$

$$\doteqdot -\sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{\Omega_\nu(-z)\cos\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)-\psi_\nu(-z)\sin\left(\frac{z}{\sqrt{2}}-\tfrac{1}{8}\pi+\tfrac{1}{2}\nu\pi\right)\right\},$$

where $\lambda_\nu(-z)$, $\chi_\nu(-z)$, $\psi_\nu(-z)$, and $\Omega_\nu(-z)$ are obtained by writing $-z$ for z in the formulae given previously at 186 and 188.

234. $\displaystyle\int^z z\operatorname{ker} kz\,dz = \frac{z}{k}\operatorname{kei}' kz$; or $\displaystyle\frac{d}{dz}(z\operatorname{kei}' kz) = kz\operatorname{ker} kz.$

235. $\displaystyle\int^z z\operatorname{kei} kz\,dz = -\frac{z}{k}\operatorname{ker}' kz$; or $\displaystyle\frac{d}{dz}(z\operatorname{ker}' kz) = -kz\operatorname{kei} kz.$

236. $\displaystyle\int^z (\operatorname{ker}_\nu^2 kz + \operatorname{kei}_\nu^2 kz)z\,dz = \frac{z}{k}(\operatorname{ker}_\nu kz\operatorname{kei}_\nu' kz - \operatorname{kei}_\nu kz\operatorname{ker}_\nu' kz).$

237. $\displaystyle\int^z (\operatorname{ker}^2 kz + \operatorname{kei}^2 kz)z\,dz = \frac{z}{k}N_0(z)N_1(z)\sin(\phi_1-\phi_0-\tfrac{1}{4}\pi).$

238. $\displaystyle\int^z (\operatorname{ker}_\nu'^2 kz + \operatorname{kei}_\nu'^2 kz)z\,dz = \frac{z}{k}(\operatorname{ker}_\nu kz\operatorname{ker}_\nu' kz + \operatorname{kei}_\nu kz\operatorname{kei}_\nu' kz).$

239. $\displaystyle\int^z (\operatorname{ker}'^2 kz + \operatorname{kei}'^2 kz)z\,dz = \frac{z}{k}N_0(z)N_1(z)\cos(\phi_1-\phi_0-\tfrac{1}{4}\pi).$

240. $\int^z (\operatorname{ber}' z\operatorname{ker}' z - \operatorname{bei}' z\operatorname{kei}' z)z\,dz = z(\operatorname{ber}_1 z\operatorname{ker}_1 z - \operatorname{bei}_1 z\operatorname{kei}_1 z).$

241. $\operatorname{ker}_\nu^2 z + \operatorname{kei}_\nu^2 z = N_\nu^2(z).$

242. $\operatorname{ker}'^2 z + \operatorname{kei}'^2 z = N_1^2(z).$

243. $\operatorname{ker} z\operatorname{kei}' z - \operatorname{kei} z\operatorname{ker}' z = N_0(z)N_1(z)\sin(\phi_1-\phi_0-\tfrac{1}{4}\pi).$

244. $\operatorname{ker} z\operatorname{ker}' z + \operatorname{kei} z\operatorname{kei}' z = N_0(z)N_1(z)\cos(\phi_1-\phi_0-\tfrac{1}{4}\pi).$

245. $\operatorname{bei}' z\operatorname{ker}' z - \operatorname{ber}' z\operatorname{kei}' z = M_1(z)N_1(z)\sin(\theta_1-\phi_1).$

246. $\operatorname{ber}' z\operatorname{ker}' z + \operatorname{bei}' z\operatorname{kei}' z = M_1(z)N_1(z)\cos(\theta_1-\phi_1).$

TABLES

TABLE 1

$J_0(z)$

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	1·0000	0·9975	0·9900	0·9776	0·9604	0·9385	0·9120	0·8812	0·8463	0·8075
1	0·7652	0·7196	0·6711	0·6201	0·5669	0·5118	0·4554	0·3980	0·3400	0·2818
2	0·2239	0·1666	0·1104	0·0555	0·0025	−0·0484	−0·0968	−0·1424	−0·1850	−0·2243
3	−0·2601	−0·2921	−0·3202	−0·3443	−0·3643	−0·3801	−0·3918	−0·3992	−0·4026	−0·4018
4	−0·3971	−0·3887	−0·3766	−0·3610	−0·3423	−0·3205	−0·2961	−0·2693	−0·2404	−0·2097
5	−0·1776	−0·1443	−0·1103	−0·0758	−0·0412	−0·0068	0·0270	+0·0599	0·0917	0·1220
6	0·1506	0·1773	0·2017	0·2238	0·2433	0·2601	0·2740	0·2851	0·2931	0·2981
7	0·3001	0·2991	0·2951	0·2882	0·2786	0·2663	0·2516	0·2346	0·2154	0·1944
8	0·1717	0·1475	0·1222	0·0960	0·0692	0·0419	0·0146	−0·0125	−0·0392	−0·0653
9	−0·0903	−0·1142	−0·1367	−0·1577	−0·1768	−0·1939	−0·2090	−0·2218	−0·2323	−0·2403
10	−0·2459	−0·2490	−0·2496	−0·2477	−0·2434	−0·2366	−0·2276	−0·2164	−0·2032	−0·1881
11	−0·1712	−0·1528	−0·1330	−0·1121	−0·0902	−0·0677	−0·0446	−0·0213	+0·0020	0·0250
12	0·0477	0·0697	0·0908	0·1108	0·1296	0·1469	0·1626	0·1766	0·1887	0·1988
13	0·2069	0·2129	0·2167	0·2183	0·2177	0·2150	0·2101	0·2032	0·1943	0·1836
14	0·1711	0·1570	0·1414	0·1245	0·1065	0·0875	0·0679	0·0476	0·0271	0·0064
15	−0·0142	−0·0346	−0·0544	−0·0736	−0·0919	−0·1092	−0·1253	−0·1401	−0·1533	−0·1650

When $z > 15{\cdot}9$,

$$J_0(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\sin(z+\tfrac{1}{4}\pi)+\frac{1}{8z}\sin(z-\tfrac{1}{4}\pi)\right\}$$

$$\frac{0{\cdot}7979}{\sqrt{z}}\left\{\sin(57{\cdot}296z+45)^\circ+\frac{1}{8z}\sin(57{\cdot}296z-45)^\circ\right\}.$$

TABLE 2

$J_1(z)$

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	0·0000	0·0499	0·0995	0·1483	0·1960	0·2423	0·2867	0·3290	0·3688	0·4059
1	0·4401	0·4709	0·4983	0·5220	0·5419	0·5579	0·5699	0·5778	0·5815	0·5812
2	0·5767	0·5683	0·5560	0·5399	0·5202	0·4971	0·4708	0·4416	0·4097	0·3754
3	0·3391	0·3009	0·2613	0·2207	0·1792	0·1374	0·0955	0·0538	0·0128	−0·0272
4	−0·0660	−0·1033	−0·1386	−0·1719	−0·2028	−0·2311	−0·2566	−0·2791	−0·2985	−0·3147
5	−0·3276	−0·3371	−0·3432	−0·3460	−0·3453	−0·3414	−0·3343	−0·3241	−0·3110	−0·2951
6	−0·2767	−0·2559	−0·2329	−0·2081	−0·1816	−0·1538	−0·1250	−0·0953	−0·0652	−0·0349
7	−0·0047	+0·0252	0·0543	0·0826	0·1096	0·1352	0·1592	0·1813	0·2014	0·2192
8	0·2346	0·2476	0·2580	0·2657	0·2708	0·2731	0·2728	0·2697	0·2641	0·2559
9	0·2453	0·2324	0·2174	0·2004	0·1816	0·1613	0·1395	0·1166	0·0928	0·0684
10	0·0435	0·0184	−0·0066	−0·0313	−0·0555	−0·0789	−0·1012	−0·1224	−0·1422	−0·1603
11	−0·1768	−0·1913	−0·2039	−0·2143	−0·2225	−0·2284	−0·2320	−0·2333	−0·2323	−0·2290
12	−0·2234	−0·2157	−0·2060	−0·1943	−0·1807	−0·1655	−0·1487	−0·1307	−0·1114	−0·0912
13	−0·0703	−0·0489	−0·0271	−0·0052	+0·0166	0·0380	0·0590	0·0791	0·0984	0·1165
14	0·1334	0·1488	0·1626	0·1747	0·1850	0·1934	0·1999	0·2043	0·2066	0·2069
15	0·2051	0·2013	0·1955	0·1879	0·1784	0·1672	0·1544	0·1402	0·1247	0·1080

When $z > 15{\cdot}9$,

$$J_1(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\sin(z-\tfrac{1}{4}\pi)+\frac{3}{8z}\sin(z+\tfrac{1}{4}\pi)\right\}$$

$$0{\cdot}7979\left\{\sin(57{\cdot}296z-45)^\circ+\frac{3}{8z}\sin(57{\cdot}296z+45)^\circ\right\}.$$

Table 3

$Y_0(z)$

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	$-\infty$	−1·534	−1·081	−0·8073	−0·6060	−0·4445	−0·3085	−0·1907	−0·0868	+0·0056
1	0·0883	0·1622	0·2281	0·2865	0·3379	0·3824	0·4204	0·4520	0·4774	0·4968
2	0·5104	0·5183	0·5208	0·5181	0·5104	0·4981	0·4813	0·4605	0·4359	0·4079
3	0·3769	0·3431	0·3071	0·2691	0·2296	0·1890	0·1477	0·1061	0·0645	0·0234
4	−0·0169	−0·0561	−0·0938	−0·1296	−0·1633	−0·1947	−0·2235	−0·2494	−0·2723	−0·2921
5	−0·3085	−0·3216	−0·3313	−0·3374	−0·3402	−0·3395	−0·3354	−0·3282	−0·3177	−0·3044
6	−0·2882	−0·2694	−0·2483	−0·2251	−0·1999	−0·1732	−0·1452	−0·1162	−0·0864	−0·0563
7	−0·0259	+0·0042	0·0339	0·0628	0·0907	0·1173	0·1424	0·1658	0·1872	0·2065
8	0·2235	0·2381	0·2501	0·2595	0·2662	0·2702	0·2715	0·2700	0·2659	0·2592
9	0·2499	0·2383	0·2245	0·2086	0·1907	0·1712	0·1502	0·1279	0·1045	0·0804
10	0·0557	0·0307	0·0056	−0·0193	−0·0437	−0·0675	−0·0904	−0·1122	−0·1326	−0·1516
11	−0·1688	−0·1843	−0·1977	−0·2091	−0·2183	−0·2252	−0·2299	−0·2322	−0·2322	−0·2298
12	−0·2252	−0·2184	−0·2095	−0·1986	−0·1858	−0·1712	−0·1551	−0·1375	−0·1187	−0·0989
13	−0·0782	−0·0569	−0·0352	−0·0134	+0·0085	+0·0301	+0·0512	0·0717	0·0913	0·1099
14	0·1272	0·1431	0·1575	0·1703	0·1812	0·1903	0·1974	0·2025	0·2056	0·2065
15	0·2055	0·2023	0·1972	0·1902	0·1813	0·1706	0·1584	0·1446	0·1295	0·1132

When $z > 15{\cdot}9$,

$$Y_0(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\sin(z-\tfrac{1}{4}\pi)-\frac{1}{8z}\sin(z+\tfrac{1}{4}\pi)\right\}$$

$$\doteqdot \frac{0{\cdot}7979}{\sqrt{z}}\left\{\sin(57{\cdot}296z-45)^\circ-\frac{1}{8z}\sin(57{\cdot}296z+45)^\circ\right\}.$$

Table 4

$Y_1(z)$

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	$-\infty$	−6·459	−3·324	−2·293	−1·781	−1·471	−1·260	−1·103	−0·9781	−0·8731
1	−0·7812	−0·6981	−0·6211	−0·5485	−0·4791	−0·4123	−0·3476	−0·2847	−0·2237	−0·1644
2	−0·1070	−0·0517	+0·0015	+0·0523	0·1005	0·1459	0·1884	0·2276	0·2635	0·2959
3	0·3247	0·3496	0·3707	0·3879	0·4010	0·4102	0·4154	0·4167	0·4141	0·4078
4	0·3979	0·3846	0·3680	0·3484	0·3260	0·3010	0·2737	0·2445	0·2136	0·1812
5	0·1479	0·1137	0·0792	0·0445	0·0101	−0·0238	−0·0568	−0·0887	−0·1192	−0·1481
6	−0·1750	−0·1998	−0·2223	−0·2422	−0·2596	−0·2741	−0·2857	−0·2945	−0·3002	−0·3029
7	−0·3027	−0·2995	−0·2934	−0·2846	−0·2731	−0·2591	−0·2428	−0·2243	−0·2039	−0·1817
8	−0·1581	−0·1331	−0·1072	−0·0806	−0·0535	−0·0262	+0·0011	+0·0280	0·0544	0·0799
9	+0·1043	0·1275	0·1491	0·1691	0·1871	0·2032	0·2171	0·2287	0·2379	0·2447
10	0·2490	0·2508	0·2502	0·2471	0·2416	0·2337	0·2236	0·2114	0·1973	0·1813
11	0·1637	0·1446	0·1243	0·1029	0·0807	0·0579	0·0348	0·0114	−0·0118	−0·0347
12	−0·0571	−0·0787	−0·0994	−0·1189	−0·1371	−0·1538	−0·1689	−0·1821	−0·1935	−0·2028
13	−0·2101	−0·2152	−0·2182	−0·2190	−0·2176	−0·2140	−0·2084	−0·2007	−0·1912	−0·1798
14	−0·1666	−0·1520	−0·1359	−0·1186	−0·1003	−0·0810	−0·0612	−0·0408	−0·0202	+0·0005
15	0·0211	0·0413	0·0609	0·0799	0·0979	0·1148	0·1305	0·1447	0·1575	0·1686

When $z > 15{\cdot}9$,

$$Y_1(z) \doteqdot \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\sin(z-\tfrac{3}{4}\pi)+\frac{3}{8z}\sin(z-\tfrac{1}{4}\pi)\right\}$$

$$\doteqdot \frac{0{\cdot}7979}{\sqrt{z}}\left\{\sin(57{\cdot}296z-135)^\circ+\frac{3}{8z}\sin(57{\cdot}296z-45)^\circ\right\}.$$

Table 5

$J_2(z)$

0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·0000	0·0012	0·0050	0·0112	0·0197	0·0306	0·0437	0·0588	0·0758	0·0946
0·1149	0·1366	0·1593	0·1830	0·2074	0·2321	0·2570	0·2817	0·3061	0·3299
0·3528	0·3746	0·3951	0·4139	0·4310	0·4461	0·4590	0·4696	0·4777	0·4832
0·4861	0·4862	0·4835	0·4780	0·4697	0·4586	0·4448	0·4283	0·4093	0·3879
0·3641	0·3383	0·3105	0·2811	0·2501	0·2178	0·1846	0·1506	0·1161	0·0813

When $0 \leqslant z < 1$, $$J_2(z) \doteqdot \frac{z^2}{8}\left(1-\frac{z^2}{12}\right).$$

$J_3(z)$

0	0·0000	0·0000	0·0002	0·0006	0·0013	0·0026	0·0044	0·0069	0·0102	0·0144
1	0·0196	0·0257	0·0329	0·0411	0·0505	0·0610	0·0725	0·0851	0·0988	0·1134
2	0·1289	0·1453	0·1623	0·1800	0·1981	0·2166	0·2353	0·2540	0·2727	0·2911
3	0·3091	0·3264	0·3431	0·3588	0·3734	0·3868	0·3988	0·4092	0·4180	0·4250
4	0·4302	0·4333	0·4344	0·4333	0·4301	0·4247	0·4171	0·4072	0·3952	0·3811

When $0 \leqslant z < 1$, $$J_3(z) \doteqdot \frac{z^3}{48}\left(1-\frac{z^2}{16}\right).$$

$J_4(z)$

0	0·0000	0·0000	0·0000	0·0000	0·0001	0·0002	0·0003	0·0006	0·0010	0·0016
1	0·0025	0·0036	0·0050	0·0068	0·0091	0·0118	0·0150	0·0188	0·0232	0·0283
2	0·0340	0·0405	0·0476	0·0556	0·0643	0·0738	0·0840	0·0950	0·1067	0·1190
3	0·1320	0·1456	0·1597	0·1743	0·1891	0·2044	0·2198	0·2353	0·2507	0·2661
4	0·2811	0·2958	0·3100	0·3236	0·3365	0·3484	0·3594	0·3693	0·3780	0·3853

When $0 \leqslant z < 1$, $$J_4(z) \doteqdot \frac{z^4}{384}\left(1-\frac{z^2}{20}\right).$$

TABLE 6

$\mathbf{H}_0(z)$

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	0·0000	0·0636	0·1268	0·1891	0·2501	0·3096	0·3669	0·4218	0·4740	0·5230
1	0·5687	0·6106	0·6486	0·6824	0·7118	0·7367	0·7570	0·7726	0·7835	0·7895
2	0·7909	0·7875	0·7796	0·7673	0·7506	0·7300	0·7054	0·6773	0·6459	0·6114
3	0·5743	0·5348	0·4934	0·4503	0·4060	0·3608	0·3151	0·2694	0·2238	0·1789
4	0·1350	0·0924	0·0515	0·0125	−0·0248	−0·0585	−0·0901	−0·1187	−0·1442	−0·1664
5	−0·1852	−0·2006	−0·2124	−0·2208	−0·2256	−0·2268	−0·2247	−0·2193	−0·2107	−0·1990
6	−0·1846	−0·1674	−0·1479	−0·1262	−0·1025	−0·0773	−0·0507	−0·0230	+0·0054	0·0343
7	0·0634	0·0923	0·1208	0·1485	0·1753	0·2009	0·2249	0·2472	0·2677	0·2860
8	0·3020	0·3156	0·3267	0·3352	0·3410	0·3442	0·3446	0·3423	0·3374	0·3299
9	0·3199	0·3075	0·2929	0·2763	0·2578	0·2375	0·2158	0·1929	0·1689	0·1441
10	0·1187	0·0931	0·0674	0·0420	0·0169	−0·0074	−0·0309	−0·0532	−0·0742	−0·0936
11	−0·1114	−0·1274	−0·1413	−0·1532	−0·1629	−0·1703	−0·1754	−0·1781	−0·1786	−0·1767
12	−0·1725	−0·1662	−0·1577	−0·1472	−0·1348	−0·1206	−0·1048	−0·0877	−0·0693	−0·0498
13	−0·0295	−0·0086	+0·0127	+0·0342	0·0557	0·0770	0·0978	0·1179	0·1372	0·1554
14	0·1724	0·1881	0·2022	0·2146	0·2252	0·2340	0·2408	0·2456	0·2484	0·2491
15	0·2477	0·2443	0·2389	0·2316	0·2225	0·2116	0·1990	0·1850	0·1696	0·1530

When $z > 15{\cdot}9$,

$$\mathbf{H}_0(z) \doteqdot Y_0(z) + \frac{2}{\pi z}$$

$$\doteqdot \frac{0{\cdot}7979}{\sqrt{z}}\left[\sin(57{\cdot}296z - 45)^\circ - \frac{1}{8z}\sin(57{\cdot}296z + 45)^\circ\right] + \frac{0{\cdot}6366}{z}$$

TABLE 7

$\mathbf{H}_1(z)$

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	0·0000	0·0021	0·0085	0·0190	0·0336	0·0522	0·0746	0·1006	0·1301	0·1628
1	0·1985	0·2368	0·2774	0·3201	0·3645	0·4103	0·4570	0·5044	0·5521	0·5997
2	0·6468	0·6930	0·7381	0·7817	0·8235	0·8632	0·9004	0·9349	0·9665	0·9950
3	1·020	1·042	1·060	1·074	1·085	1·092	1·095	1·094	1·089	1·081
4	1·070	1·055	1·037	1·016	0·9921	0·9660	0·9376	0·9073	0·8754	0·8421
5	0·8078	0·7728	0·7375	0·7021	0·6670	0·6324	0·5987	0·5661	0·5350	0·5056
6	0·4782	0·4529	0·4299	0·4095	0·3917	0·3768	0·3647	0·3556	0·3495	0·3464
7	0·3463	0·3492	0·3549	0·3634	0·3746	0·3883	0·4044	0·4226	0·4428	0·4647
8	0·4881	0·5128	0·5385	0·5649	0·5918	0·6190	0·6460	0·6728	0·6989	0·7243
9	0·7485	0·7715	0·7930	0·8128	0·8307	0·8466	0·8604	0·8719	0·8810	0·8876
10	0·8918	0·8935	0·8928	0·8895	0·8839	0·8760	0·8658	0·8535	0·8392	0·8232
11	0·8055	0·7863	0·7659	0·7444	0·7222	0·6993	0·6760	0·6526	0·6293	0·6063
12	0·5839	0·5621	0·5414	0·5218	0·5035	0·4868	0·4717	0·4584	0·4470	0·4376
13	0·4302	0·4251	0·4220	0·4212	0·4226	0·4260	0·4316	0·4392	0·4488	0·4601
14	0·4732	0·4878	0·5038	0·5211	0·5394	0·5586	0·5784	0·5987	0·6193	0·6400
15	0·6605	0·6807	0·7003	0·7192	0·7371	0·7540	0·7697	0·7839	0·7966	0·8077

When $z > 15{\cdot}9$,

$$\mathbf{H}_1(z) \doteqdot Y_1(z) + \frac{2}{\pi}\left\{1 + \frac{1}{z^2}\right\}$$

$$\doteqdot \frac{0{\cdot}7979}{\sqrt{z}}\left\{\sin(57{\cdot}296z - 135)^\circ + \frac{3}{8z}\sin(57{\cdot}296z - 45)^\circ\right\} + 0{\cdot}6366\left(1 + \frac{1}{z^2}\right).$$

Table 8

Ber z

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	1·0000	1·0000	1·0000	0·9999	0·9996	0·9990	0·9980	0·9962	0·9936	0·9898
1	0·9844	0·9771	0·9676	0·9554	0·9401	0·9211	0·8979	0·8700	0·8367	0·7975
2	0·7517	0·6987	0·6377	0·5680	0·4890	0·4000	0·3001	0·1887	0·0651	−0·0714
3	−0·2214	−0·3855	−0·5644	−0·7584	−0·9680	−1·194	−1·435	−1·693	−1·967	−2·258
4	−2·563	−2·884	−3·219	−3·568	−3·928	−4·299	−4·678	−5·064	−5·453	−5·843
5	−6·230	−6·611	−6·980	−7·334	−7·667	−7·974	−8·247	−8·479	−8·664	−8·794
6	−8·858	−8·849	−8·756	−8·569	−8·276	−7·867	−7·329	−6·649	−5·816	−4·815
7	−3·633	−2·257	−0·6737	+1·131	3·169	5·455	7·999	10·81	13·91	17·29
8	20·97	24·96	29·25	33·84	38·74	43·94	49·42	55·19	61·21	67·47
9	73·94	80·58	87·35	94·21	101·1	108·0	114·7	121·3	127·5	133·4
10	138·8									

When $z > 10$,

$$\operatorname{ber} z \doteqdot \frac{0\cdot3989e^{z/\sqrt{2}}}{\sqrt{z}}\left\{\sin(40\cdot514z+67\cdot5)^\circ+\frac{1}{8z}\sin(40\cdot514z+22\cdot5)^\circ\right\}.$$

When $0 \leqslant z < 1$, see (6 a) p. 120.

Bei z

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	0·0000	0·0025	0·0100	0·0225	0·0400	0·0625	0·0900	0·1224	0·1599	0·2023
1	0·2496	0·3017	0·3587	0·4204	0·4867	0·5576	0·6327	0·7120	0·7953	0·8821
2	0·9723	1·065	1·161	1·259	1·357	1·457	1·557	1·656	1·753	1·847
3	1·938	2·023	2·102	2·172	2·233	2·283	2·320	2·341	2·345	2·330
4	2·293	2·231	2·142	2·024	1·873	1·686	1·461	1·195	0·8837	0·5251
5	0·1160	−0·3467	−0·8658	−1·444	−2·085	−2·789	−3·560	−4·399	−5·307	−6·285
6	−7·335	−8·454	−9·644	−10·90	−12·22	−13·61	−15·05	−16·54	−18·07	−19·64
7	−21·24	−22·85	−24·46	−26·05	−27·61	−29·12	−30·55	−31·88	−33·09	−34·15
8	−35·02	−35·67	−36·06	−36·16	−35·92	−35·30	−34·25	−32·71	−30·65	−28·00
9	−24·71	−20·72	−15·98	−10·41	−3·969	3·411	11·79	21·22	31·76	43·46
10	56·37									

When $z > 10$,

$$\operatorname{bei} z \doteqdot \frac{0\cdot3989e^{z/\sqrt{2}}}{\sqrt{z}}\left\{\sin(40\cdot514z-22\cdot5)^\circ+\frac{1}{8z}\sin(40\cdot514z-67\cdot5)^\circ\right\}.$$

When $0 \leqslant z < 1$, see (6 b), p. 120.

TABLE 9

Ber′z

0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·0000	0·0000	−0·0005	−0·0017	−0·0040	−0·0078	−0·0135	−0·0214	−0·0320	−0·0455
−0·0624	−0·0831	−0·1078	−0·1370	−0·1709	−0·2100	−0·2545	−0·3048	−0·3612	−0·4238
−0·4931	−0·5691	−0·6520	−0·7420	−0·8392	−0·9436	−1·055	−1·174	−1·299	−1·431
−1·570	−1·714	−1·864	−2·018	−2·175	−2·336	−2·498	−2·661	−2·822	−2·981
−3·135	−3·282	−3·420	−3·547	−3·659	−3·754	−3·828	−3·878	−3·901	−3·891
−3·845	−3·759	−3·627	−3·445	−3·206	−2·907	−2·541	−2·102	−1·586	−0·9844
−0·2931	+0·4943	1·384	2·380	3·490	4·717	6·067	7·544	9·151	10·89
12·76	14·77	16·92	19·19	21·60	24·13	26·78	29·53	32·38	35·31
38·31	41·35	44·42	47·47	50·49	53·44	56·28	58·97	61·45	63·68
65·60	67·14	68·25	68·83	68·82	68·13	66·67	64·35	61·07	56·72
51·20									

When $z > 10$,

$$\operatorname{ber}'z \doteqdot \frac{0{\cdot}3989e^{z/\sqrt{2}}}{\sqrt{z}}\left\{\sin(40{\cdot}514z+112{\cdot}5)^\circ-\frac{3}{8z}\sin(40{\cdot}514z+67{\cdot}5)^\circ\right\}.$$

When $0 \leqslant z < 1$, differentiate (6a) p. 120.

Bei′z

z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0	0·0000	0·0500	0·1000	0·1500	0·2000	0·2499	0·2998	0·3496	0·3991	0·4485
1	0·4974	0·5458	0·5935	0·6403	0·6860	0·7303	0·7727	0·8131	0·8509	0·8857
2	0·9170	0·9442	0·9666	0·9836	0·9944	0·9983	0·9943	0·9815	0·9590	0·9257
3	0·8805	0·8223	0·7499	0·6621	0·5577	0·4353	0·2937	0·1315	−0·0525	−0·2597
4	−0·4911	−0·7481	−1·032	−1·343	−1·683	−2·053	−2·452	−2·882	−3·342	−3·833
5	−4·354	−4·905	−5·484	−6·089	−6·720	−7·373	−8·045	−8·734	−9·433	−10·14
6	−10·85	−11·55	−12·23	−12·90	−13·54	−14·13	−14·67	−15·15	−15·54	−15·85
7	−16·04	−16·11	−16·03	−15·79	−15·37	−14·74	−13·88	−12·76	−11·37	−9·681
8	−7·660	−5·285	−2·530	+0·6341	4·232	8·290	12·83	17·88	23·47	29·60
9	36·30	43·58	51·46	59·94	69·01	78·68	88·94	99·76	111·1	123·0
10	135·3									

When $z > 10$,

$$\operatorname{bei}'z \doteqdot \frac{0{\cdot}3989e^{z/\sqrt{2}}}{\sqrt{z}}\left\{\sin(40{\cdot}514z+22{\cdot}5)^\circ-\frac{3}{8z}\sin(40{\cdot}514z-22{\cdot}5)^\circ\right\}.$$

When $0 \leqslant z < 1$, differentiate (6b) p. 120.

TABLE 10

Ker z

	z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
	0	$+\infty$	2·420	1·733	1·337	1·063	0·8559	0·6931	0·5614	0·4529	0·3625
	1	0·2867	0·2228	0·1689	0·1235	0·0851	0·0529	0·0260	0·0037	−0·0147	−0·0297
	2	−0·0417	−0·0511	−0·0583	−0·0637	−0·0674	−0·0697	−0·0708	−0·0710	−0·0703	−0·0689
	3	−0·0670	−0·0647	−0·0620	−0·0590	−0·0559	−0·0526	−0·0493	−0·0460	−0·0426	−0·0394
$10^{-5}\times$	4	−3,618	−3,308	−3,011	−2,726	−2,456	−2,200	−1,960	−1,734	−1,525	−1,330
$10^{-5}\times$	5	−1,151	−986·5	−835·9	−698·9	−574·9	−463·2	−363·2	−274·0	−195·2	−125·8
$10^{-6}\times$	6	−653·0	−129·5	+319·1	699·1	1,017	1,278	1,488	1,653	1,777	1,866
$10^{-6}\times$	7	1,922	1,951	1,956	1,940	1,907	1,860	1,800	1,731	1,655	1,572
$10^{-6}\times$	8	1,486	1,397	1,306	1,216	1,126	1,037	951·1	867·5	787·1	710·2
$10^{-6}\times$	9	637·2	568·1	503·0	442·2	385·5	333·0	284·6	240·2	199·6	162·8
$10^{-6}\times$	10	129·5									

When $z > 10$,

$$\ker z \doteqdot \frac{1{\cdot}2533e^{-z/\sqrt{2}}}{\sqrt{z}}\left\{\sin(40{\cdot}514z+112{\cdot}5)^\circ+\frac{1}{8z}\sin(40{\cdot}514z-22{\cdot}5)^\circ\right\}.$$

When $0 \leqslant z < 1$,

$$\ker z = A_0+\frac{1}{4}\pi v^2-\frac{1}{4}\left(A_0+\frac{3}{2}\right)v^4-\frac{1}{144}\pi v^6+\frac{1}{576}\left(A_0+\frac{25}{12}\right)v^8-\dots .$$

Kei z

	z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
	0	−0·7854	−0·7769	−0·7581	−0·7331	−0·7038	−0·6716	−0·6374	−0·6022	−0·5664	−0·5305
	1	−0·4950	−0·4601	−0·4262	−0·3933	−0·3617	−0·3314	−0·3026	−0·2752	−0·2494	−0·2251
	2	−0·2024	−0·1812	−0·1614	−0·1431	−0·1262	−0·1107	−0·0964	−0·0834	−0·0716	−0·0608
$10^{-5}\times$	3	−5,112	−4,240	−3,458	−2,762	−2,145	−1,600	−1,123	−707·7	−348·7	−41·08
$10^{-6}\times$	4	+2,198	4,386	6,194	7,661	8,826	9,721	10,380	10,830	11,100	11,210
$10^{-6}\times$	5	11,190	11,050	10,820	10;510	10,140	9,716	9,255	8,766	8,258	7,739
$10^{-6}\times$	6	7,216	6,696	6,183	5,681	5,194	4,724	4,274	3,846	3,440	3,058
$10^{-6}\times$	7	2,700	2,366	2,057	1,770	1,507	1,267	1,048	849·8	671·4	511·7
$10^{-6}\times$	8	369·6	244·0	133·9	+38·09	−44·49	−114·9	−174·1	−223·3	−263·2	−294·9
$10^{-6}\times$	9	−319·2	−336·8	−348·6	−355·2	−357·4	−355·7	−350·8	−343·0	−332·9	−321·0
$10^{-6}\times$	10	−307·5									

When $z > 10$,

$$\operatorname{kei} z \doteqdot \frac{1{\cdot}2533e^{-z/\sqrt{2}}}{\sqrt{z}}\left\{-\sin(40{\cdot}514z+22{\cdot}5)+\frac{1}{8z}\sin(40{\cdot}514z+67{\cdot}5)^\circ\right\}.$$

When $0 \leqslant z < 1$,

$$\operatorname{kei} z = -\frac{1}{4}\pi+\left(A_0+1\right)v^2+\frac{1}{16}\pi v^4-\frac{1}{36}\left(A_0+\frac{11}{6}\right)v^6-\frac{1}{2304}\pi v^8,$$

where $v = \frac{1}{2}z$, and $A_0 = (0{\cdot}1159315\dots-\log_e z)$.

Table 11

Ker′z

	z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
	0	$-\infty$	−9·961	−4·923	−3·220	−2·352	−1·820	−1·457	−1·191	−0·9873	−0·8259
	1	−0·6946	−0·5859	−0·4946	−0·4172	−0·3511	−0·2942	−0·2451	−0·2027	−0·1659	−0·1341
$10^{-4}\times$	2	−1,066	−828·2	−623·4	−447·5	−297·1	−169·3	−61·36	+29·04	+104·0	165·3
$10^{-4}\times$	3	214·8	253·7	283·6	305·6	320·7	329·9	334·1	334·0	330·4	323·8
$10^{-4}\times$	4	314·8	303·8	291·3	277·7	263·2	248·1	232·8	217·3	201·9	186·8
$10^{-4}\times$	5	171·9	157·5	143·7	130·4	117·7	105·8	94·47	83·88	74·00	64·81
$10^{-4}\times$	6	56·32	48·50	41·33	34·79	28·85	23·49	18·67	14·36	10·54	7·164
$10^{-6}\times$	7	420·5	163·3	−58·39	−247·4	−406·6	−538·8	−646·5	−732·2	−798·2	−846·7
$10^{-6}\times$	8	−879·7	−899·2	−906·9	−904·4	−893·2	−874·7	−850·0	−820·4	−786·8	−750·2
$10^{-6}\times$	9	−711·2	−670·7	−629·3	−587·5	−545·8	−504·5	−464·1	−424·8	−386·8	−350·4
$10^{-6}\times$	10	−315·6									

When $z > 10$,

$$\operatorname{ker}'z \doteqdot -\frac{1\cdot2533e^{-z/\sqrt{2}}}{\sqrt{z}}\left\{\sin(40\cdot514z+67\cdot5)^\circ+\frac{3}{8z}\sin(40\cdot514z+112\cdot5)^\circ\right\}.$$

When $0 \leqslant z < 1$,

$$\operatorname{ker}'z \doteqdot -\frac{1}{2v}+\frac{1}{4}\pi v-\frac{1}{2}\left(A_0+\frac{5}{4}\right)v^3-\frac{1}{48}\pi v^5+\frac{1}{144}\left(A_0+\frac{47}{24}\right)v^7.$$

Kei′z

	z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
	0	0·0000	0·1460	0·2229	0·2743	0·3095	0·3332	0·3482	0·3563	0·3590	0·3574
	1	0·3524	0·3445	0·3345	0·3227	0·3096	0·2956	0·2809	0·2658	0·2504	0·2351
	2	0·2198	0·2048	0·1901	0·1759	0·1621	0·1489	0·1363	0·1243	0·1129	0·1021
$10^{-5}\times$	3	9,204	8,259	7,378	6,558	5,799	5,098	4,454	3,864	3,325	2,835
$10^{-5}\times$	4	2,391	1,991	1,631	1,310	1,024	771·5	549·2	355·0	186·5	41·52
$10^{-6}\times$	5	−820·0	−1,861	−2,726	−3,433	−4,000	−4,440	−4,769	−5,000	−5,146	−5,217
$10^{-6}\times$	6	−5,224	−5,176	−5,082	−4,951	−4,788	−4,600	−4,393	−4,170	−3,939	−3,701
$10^{-6}\times$	7	−3,460	−3,218	−2,979	−2,745	−2,517	−2,296	−2,084	−1,881	−1,689	−1,507
$10^{-6}\times$	8	−1,336	−1,177	−1,028	−890·2	−763·2	−646·7	−540·4	−443·8	−356·5	−278·1
$10^{-6}\times$	9	−208·1	−145·9	−91·09	−43·15	−1·559	+34·16	64·49	89·89	110·8	127·7
$10^{-6}\times$	10	140·9									

When $z > 10$,

$$\operatorname{kei}'z \doteqdot \frac{1\cdot2533e^{-z/\sqrt{2}}}{\sqrt{z}}\left\{\sin(40\cdot514z-22\cdot5)^\circ+\frac{3}{8z}\sin(40\cdot514z+22\cdot5)^\circ\right\}.$$

When $0 \leqslant z < 1$,

$$\operatorname{kei}'z \doteqdot \left(A_0+\frac{1}{2}\right)v+\frac{1}{8}\pi v^3-\frac{1}{12}\left(A_0+\frac{5}{3}\right)v^5-\frac{1}{576}\pi v^7,$$

where $v = \frac{1}{2}z$, and $A_0 = (0\cdot1159315\ldots-\log_e z)$.

Table 12

$\mathrm{Ber}_n z$, $\mathrm{bei}_n z$, $\mathrm{ber}'_n z$, and $\mathrm{bei}'_n z$, from $n = 1$ to 5

z	1	2	3	4	5	6	7	8	9	10
$\mathrm{ber}_1 z$	0·3959	0·9971	1·783	−1·869	0·3598	7·462	20·37	32·51	20·72	−59·48
$\mathrm{bei}_1 z$	+0·3076	0·2998	0·4875	−2·564	−5·798	−7·877	−2·317	21·67	72·05	131·9
$\mathrm{ber}'_1 z$	0·4767	0·7205	−0·6360	0·6587	4·251	10·21	14·68	5·866	−37·11	−132·1
$\mathrm{bei}'_1 z$	+0·2120	−0·3058	−1·364	−2·793	−3·328	0·2355	12·78	36·88	61·75	45·13
$\mathrm{ber}_2 z$	0·0104	0·1653	0·8084	2·318	4·488	5·243	−0·9504	−22·89	−65·87	−111·8
$\mathrm{bei}_2 z$	0·1247	0·4792	0·8910	−0·7254	1·422	7·432	17·59	25·44	10·13	−66·61
$\mathrm{ber}'_2 z$	+0·0416	0·3278	1·031	1·976	2·050	−1·455	−12·49	−32·59	−50·96	−28·84
$\mathrm{bei}'_2 z$	0·2480	0·4378	−0·2865	0·8538	3·785	8·369	11·02	1·301	−38·55	−122·0
$\mathrm{ber}_3 z$	+0·0138	0·0856	0·1304	−0·2826	−2·094	−6·430	−12·88	−15·42	3·167	72·25
$\mathrm{bei}_3 z$	0·0156	0·1442	0·5654	1·438	2·454	+1·901	−4·407	−22·58	−54·54	−81·42
$\mathrm{ber}'_3 z$	0·0394	0·0936	0·0720	−0·9141	−2·923	−5·748	−6·249	3·980	38·35	104·5
$\mathrm{bei}'_3 z$	0·0486	0·2394	0·6363	1·074	0·6956	−2·499	−11·22	−25·71	−35·56	−7·513
$\mathrm{ber}_4 z$	0·0026	−0·0410	−0·1933	−0·4931	−0·6287	0·6483	6·084	19·09	38·67	46·58
$\mathrm{bei}_4 z$	0·00013	0·0083	−0·0930	0·4999	−1·7276	−4·230	−7·117	−5·289	14·08	70·50
$\mathrm{ber}'_4 z$	0·0104	−0·0806	−0·2343	−0·3237	0·2483	2·770	8·745	17·32	19·14	−12·15
$\mathrm{bei}'_4 z$	−0·00078	−0·0248	−0·1835	−0·7167	−1·834	−3·071	−1·922	7·704	34·55	80·47
$\mathrm{ber}_5 z$	+0·00019	0·0068	0·0586	0·2731	0·8510	1·831	2·209	−1·821	−18·62	−58·72
$\mathrm{bei}_5 z$	0·00018	0·0048	−0·0255	−0·0335	0·2114	1·476	5·242	12·81	21·38	15·19
$\mathrm{ber}'_5 z$	0·00097	+0·0178	0·1048	0·3608	0·8151	1·007	−0·8472	−8·624	−26·96	−53·43
$\mathrm{bei}'_5 z$	0·00087	−0·0110	−0·0283	0·0467	0·5656	2·220	5·590	9·234	+ 5·504	−24·51

Table 13

$\mathrm{Ker}_n z$, $\mathrm{kei}_n z$, $\mathrm{ker}'_n z$, and $\mathrm{kei}'_n z$, from $n = 1$ to 5

z	1	2	3	4	5	6	7	8	9	10
$\mathrm{ker}_1 z$	7,403	2,308	−499·0	53·51	127·4	76·76	27·44	3·229	−3·558	−3·228
$\mathrm{kei}_1 z$	2,420	800·5	802·7	391·7	115·8	2·884	−21·49	−15·67	−6·501	−1·235
$\mathrm{ker}'_1 z$	8,876	2,880	1,002	226·9	−23·18	−59·20	−36·60	−13·52	−1·853	+1·582
$\mathrm{kei}'_1 z$	7,947	736·3	−380·1	−369·3	−183·7	−56·13	1·566	9·852	7·485	3·214

Multiply all values by 10^{-4}

z	1	2	3	4	5	6	7	8	9	10
$\mathrm{ker}_2 z$	4,180	2,615	1,284	481·3	111·8	−10·88	−29·10	−18·20	−6·834	−1·013
$\mathrm{kei}_2 z$	18,842	3,090	368·0	−179·4	−180·6	−90·94	−28·21	−1·497	4·772	3·706
$\mathrm{ker}'_2 z$	1,415	−1,549	−1,071	−555·5	−216·7	−52·69	4·111	13·35	8·631	3·359
$\mathrm{kei}'_2 z$	41,208	−5,288	−1,166	−149·4	80·46	82·55	42·65	13·74	1·020	−2·150

Multiply all values by 10^{-4}

z	1	2	3	4	5	6	7	8	9	10
$\mathrm{ker}_3 z$	48,873	2,980	−364·5	−520·7	−292·8	−114·5	−27·07	2·677	7·205	4·563
$\mathrm{kei}_3 z$	62,697	−8,868	−2,360	−605·2	−76·85	45·12	44·65	22·63	7·148	0·473
$\mathrm{ker}'_3 z$	−16,290	−850·4	−80·36	17·70	22·44	12·92	5·213	1·292	−0·0944	−0·3273
$\mathrm{kei}'_3 z$	17,772	1,297	300·8	92·11	25·29	3·405	−1·977	−2·030	−1·059	−0·3479

Multiply all values of $\mathrm{ker}_3 z$ and $\mathrm{kei}_3 z$ by 10^{-4} and those of $\mathrm{ker}'_3 z$ and $\mathrm{kei}'_3 z$ by 10^{-3}

z	1	2	3	4	5	6	7	8	9	10
$\mathrm{ker}_4 z$	−47,753	−2,775	−410·6	−57·09	7·143	12·38	7·257	2·878	0·6807	−0·0722
$\mathrm{kei}_4 z$	3,981	940·0	348·5	137·4	49·43	14·00	1·780	−1·193	−1·154	−0·5843
$\mathrm{ker}'_4 z$	192,000	5,966	740·2	136·7	20·43	−3·344	−5·361	−3·229	−1·318	−0·3272
$\mathrm{kei}'_4 z$	−8,035	−1,042	−323·6	−131·4	−54·82	−20·62	−6·088	−0·8148	0·5168	0·5229

Multiply all values by 10^{-3}

z	1	2	3	4	5	6	7	8	9	10
$\mathrm{ker}_5 z$	287·8	10·21	1·468	327·1×	77·13×	12·98×	−1·719×	−3·146×	−1·874×	−0·7460×
$\mathrm{kei}_5 z$	253·9	6·077	0·3531	−52·99×	−56·32×	−29·38×	−11·77×	−3·455×	−0·4175×	0·3241×
$\mathrm{ker}'_5 z$	−1,408	−24·23	−2·403	−465·6×	−117·1×	−29·47×	−5·162×	0·7744×	1·375×	0·8372×
$\mathrm{kei}'_5 z$	−1,306	−17·82	−1·125	−71·26×	26·42×	23·33×	11·28×	5·038×	1·529×	0·2001×

Multiply values marked × by 10^{-3}

Table 14

$$J_0(zi^{\frac{3}{2}}) = M_0(z)e^{i\theta_0(z)} = \operatorname{ber} z + i \operatorname{bei} z$$

z	$M_0(z)$	$\log\{\sqrt{z}M_0(z)\}$	$\theta_0(z)$	z	$M_0(z)$	$\log\{\sqrt{z}M_0(z)\}$	$\theta_0(z)$
0·00	1·000	..	0·00°	2·0	1·229	0·2401	52·29°
0·05	1·000	$\bar{1}$·3995	0·04	2·1	1·274	0·2663	56·74
0·10	1·000	$\bar{1}$·5000	0·14	2·2	1·325	0·2933	61·22
0·15	1·000	$\bar{1}$·5880	0·32	2·3	1·381	0·3210	65·71
0·20	1·000	$\bar{1}$·6505	0·57	2·4	1·443	0·3493	70·19
0·25	1·000	$\bar{1}$·6990	0·90°	2·5	1·511	0·3783	74·65°
0·30	1·000	$\bar{1}$·7386	1·29	2·6	1·586	0·4077	79·09
0·35	1·000	$\bar{1}$·7721	1·75	2·7	1·666	0·4375	83·50
0·40	1·000	$\bar{1}$·8012	2·29	2·8	1·754	0·4676	87·87
0·45	1·001	$\bar{1}$·8269	2·90	2·9	1·849	0·4980	92·21
0·50	1·001	$\bar{1}$·8499	3·58°	3·0	1·950	0·5286	96·52°
0·55	1·001	$\bar{1}$·8708	4·33	3·1	2·059	0·5594	100·79
0·60	1·002	$\bar{1}$·8900	5·15	3·2	2·176	0·5902	105·03
0·65	1·003	$\bar{1}$·9077	6·04	3·3	2·301	0·6212	109·25
0·70	1·004	$\bar{1}$·9242	7·01	3·4	2·434	0·6521	113·43
0·75	1·005	$\bar{1}$·9397	8·04°	3·5	2·576	0·6830	117·60°
0·80	1·006	$\bar{1}$·9543	9·14	3·6	2·728	0·7140	121·75
0·85	1·008	$\bar{1}$·9682	10·31	3·7	2·889	0·7449	125·87
0·90	1·010	$\bar{1}$·9815	11·55	3·8	3·061	0·7758	129·99
0·95	1·013	$\bar{1}$·9943	12·86	3·9	3·244	0·8067	134·10
1·00	1·016	0·0067	14·23°	4·0	3·439	0·8375	138·19°
1·05	1·019	0·0187	15·66	4·5	4·618	0·9910	158·59
1·10	1·023	0·0304	17·16	5·0	6·231	1·1441	178·93
1·15	1·027	0·0419	18·72	5·5	8·447	1·2969	199·28
1·20	1·032	0·0533	20·34	6·0	11·50	1·4498	219·62
1·25	1·038	0·0645	22·02°	7·0	21·55	1·7560	260·29°
1·30	1·044	0·0756	23·75	8·0	40·82	2·0624	300·92
1·35	1·051	0·0867	25·54	9·0	77·96	2·3690	341·52
1·40	1·059	0·0978	27·37	10·0	149·8	2·6756	382·10
1·45	1·067	0·1089	29·26	11·0	289·5	2·9824	422·66
1·50	1·077	0·1201	31·19°	12·0	561·8	3·2892	463·22°
1·55	1·087	0·1314	33·16	14·0	2,137	3·9029	544·32
1·60	1·890	0·1428	35·17	16·0	8,217	4·5168	625·40
1·65	1·111	0·1544	37·22	18·0	$3{,}185_1$	5·1307	706·46
1·70	1·124	0·1661	39·30	20·0	$1{,}242_2$	5·7447	787·52
1·75	1·139	0·1779	41·41°	25·0	$3{,}809_3$	7·2798	990·15°
1·80	1·154	0·1900	43·54	30·0	$1{,}192_5$	8·8150	1,192·75
1·85	1·171	0·2022	45·70	35·0	$3{,}786_6$	10·3502	1,395·35
1·90	1·189	0·2146	47·88	40·0	$1{,}215_8$	11·8856	1,597·94
1·95	1·208	0·2273	50·08	45·0	$3{,}929_9$	13·4209	1,800·53

($1{,}215_8$ represents $1{,}215 \times 10^8$.)

When $z > 45$, $M_0(z)$ and $\theta_0(z)$ can be found to 4 decimal places and to the nearest 0·001°, respectively, from the formulae:

$$\log_{10} M_0(z) = 0\cdot307093z + \frac{0\cdot0384}{z} - 0\cdot39909 - \tfrac{1}{2}\log_{10} z,$$

$$\theta_0(z) = 40\cdot51423z - \frac{5\cdot06}{z} - 22\cdot5 \text{ (degrees).}$$

The function $\log\{\sqrt{z}M_0(z)\}$ is tabulated in case it is desired to interpolate. Proportional parts may be used for interpolation without introducing an error $>$ 1 in the last figure, except for $M_0(z)$ when $z > 2\cdot7$ and for $\log\{\sqrt{z}M_0(z)\}$ when $z < 0\cdot75$. When $z > 2\cdot7$, $\log\{\sqrt{z}M_0(z)\}$ should be found from the table and $M_0(z)$ can be deduced therefrom. See also the formulae in examples 59, 60, p. 133.

TABLE 15

$$J_1(zi^{\frac{3}{2}}) = M_1(z)e^{i\theta_1(z)} = \mathrm{ber}_1 z + i\,\mathrm{bei}_1 z$$

z	$M_1(z)$	$\log\{\sqrt{z}M_1(z)\}$	$\theta_1(z)$	z	$M_1(z)$	$\log\{\sqrt{z}M_1(z)\}$	$\theta_1(z)$
0·00	0·0000		135·00°	2·25	1·199	0·2548	170·50°
0·05	0·0250	$\bar{3}$·7474	135·02	2·30	1·232	0·2715	172·03
0·10	0·0500	$\bar{2}$·1990	135·07	2·35	1·266	0·2881	173·58
0·15	0·0750	$\bar{2}$·4631	135·16	2·40	1·301	0·3045	175·16
0·20	0·1000	$\bar{2}$·6505	135·29	2·45	1·337	0·3207	176·76
0·25	0·1250	$\bar{2}$·7959	135·45°	2·50	1·374	0·3368	178·39°
0·30	0·1500	$\bar{2}$·9147	135·64	2·55	1·411	0·3529	180·03
0·35	0·1750	$\bar{1}$·0151	135·88	2·60	1·450	0·3688	181·70
0·40	0·2000	$\bar{1}$·1021	136·15	2·65	1·489	0·3846	183·39
0·45	0·2250	$\bar{1}$·1788	136·45	2·70	1·530	0·4004	185·10
0·50	0·2500	$\bar{1}$·2475	136·79°	2·80	1·615	0·4317	188·57°
0·55	0·2751	$\bar{1}$·3096	137·17	2·90	1·705	0·4628	192·11
0·60	0·3001	$\bar{1}$·3663	137·58	3·00	1·800	0·4938	195·71
0·65	0·3252	$\bar{1}$·4185	138·03	3·10	1·901	0·5247	199·37
0·70	0·3502	$\bar{1}$·4669	138·51	3·20	2·009	0·5555	203·08
0·75	0·3753	$\bar{1}$·5119	139·03°	3·30	2·124	0·5863	206·83°
0·80	0·4004	$\bar{1}$·5541	139·58	3·40	2·246	0·6171	210·62
0·85	0·4256	$\bar{1}$·5937	140·17	3·50	2·376	0·6479	214·44
0·90	0·4508	$\bar{1}$·6311	140·80	3·60	2·515	0·6788	218·30
0·95	0·4260	$\bar{1}$·6665	141·46	3·70	2·664	0·7096	222·17
1·00	0·5013	$\bar{1}$·7001	142·16°	3·80	2·823	0·7405	226·07°
1·05	0·5267	$\bar{1}$·7321	142·89	4·00	3·173	0·8025	233·90
1·10	0·5521	$\bar{1}$·7627	143·66	4·25	3·681	0·8801	243·77
1·15	0·5776	$\bar{1}$·7920	144·46	4·50	4·278	0·9579	253·67
1·20	0·6032	$\bar{1}$·8201	145·29	5·00	5·809	1·1136	273·55
1·25	0·6290	$\bar{1}$·8471	146·17°	5·5	7·925	1·2692	293·48°
1·30	0·6548	$\bar{1}$·8731	147·07	6·0	10·85	1·4245	313·45
1·35	0·6808	$\bar{1}$·8982	148·02	6·5	14·90	1·5795	333·46
1·40	0·7070	$\bar{1}$·9225	148·99	7·0	20·50	1·7343	353·51
1·45	0·7333	$\bar{1}$·9460	150·00	7·5	28·27	1·8889	373·59
1·50	0·7598	$\bar{1}$·9688	151·04°	8·0	39·07	2·0434	393·69°
1·55	0·7866	$\bar{1}$·9909	152·12	9·0	74·97	2·3520	433·96
1·60	0·8136	0·0125	153·23	10·0	144·7	2·6604	474·28
1·65	0·8408	0·0335	154·38	11·0	280·4	2·9685	514·63
1·70	0·8684	0·0539	155·55	12·0	545·6	3·2765	555·02
1·75	0·8962	0·0739	156·76°	14·0	2,084	3·8920	635·84°
1·80	0·9244	0·0935	158·00	16·0	8,038	4·5072	716·72
1·85	0·9530	0·1127	159·27	18·0	$3{,}123_1$	5·1222	797·63
1·90	0·9819	0·1315	160·57	20·0	$1{,}220_2$	5·7370	878·57
1·95	1·011	0·1499	161·90	25·0	$3{,}755_3$	7·2736	1,080·98
2·00	1·041	0·1680	163·27°	30·0	$1{,}178_5$	8·8099	1,283·45°
2·05	1·072	0·1859	164·66	35·0	$3{,}748_6$	10·3459	1,485·94
2·10	1·102	0·2035	166·08	40·0	$1{,}204_8$	11·8817	1,688·46
2·15	1·134	0·2208	167·53	45·0	$3{,}899_9$	13·4175	1,890·98
2·20	1·166	0·2379	169·00	50·0	$1{,}270_{11}$	14·9532	2,093·52

($1{,}178_5$ represents $1{,}178 \times 10^5$.)

When $z > 50$, $M_1(z)$ and $\theta_1(z)$ can be found to 4 decimal places and to the nearest 0·001°, respectively, from the formulae:

$$\log_{10} M_1(z) = 0{\cdot}307093z - \frac{0{\cdot}1152}{z} - 0{\cdot}39909 - \tfrac{1}{2}\log_{10} z,$$

$$\theta_1(z) = 40{\cdot}51423z + \frac{15{\cdot}19}{z} + 67{\cdot}5 \text{ (degrees)}.$$

The function $\log\{\sqrt{z}M_1(z)\}$ is tabulated in case it is desired to interpolate. Proportional parts may be used for interpolation, without introducing an error > 1 in the last figure, except for $M_1(z)$ when $z > 3{\cdot}3$ and for $\log\{\sqrt{z}M_1(z)\}$ when $z < 1{\cdot}3$. When $z > 3{\cdot}3$, $\log\{\sqrt{z}M_1(z)\}$ should be found from the table and $M_1(z)$ deduced therefrom. See also the formulae in examples 61, 62, p. 133.

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INDEX

accession to inertia, flexible disk, 91.
— — membrane, 93.
— — rigid disk, 84.
— — sphere, 35, 40.
acoustical impedance of horn, 88.
— power factor, 74, 77.
— pressure on rigid disk, 54, 80, 81.
admittance of transmission line, 107.
alternating current resistance, of straight wire, 140.
— — — of solenoid with core, 145.
angular range for complex variable, alteration of, 61–3.
— — — — symbolism for, x.
application of ber and bei functions, 134–9.
— of ker and kei functions, 149.
artificial line for balancing cable, 106.
asymptotic expansion, definition, 70.
— — ber z, bei z, 177.
— — $\text{ber}_\nu z$, $\text{bei}_\nu z$, 169.
— — $\text{ber}'z$, $\text{bei}'z$, 178.
— — $\text{ber}'_\nu z$, $\text{bei}'_\nu z$, 169.
— — $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, 71, 161, 162.
— — $\mathbf{H}_0(z)$, $\mathbf{H}_1(z)$, 176.
— — $\mathbf{H}_\nu(z)$, 166.
— — $I_\nu(z)$, 163.
— — $J_0(z)$, 85, 157, 173.
— — $J_\nu(z)$, 69, 158.
— — $K_0(z)$, 164.
— — $K_\nu(z)$, 165.
— — ker z, kei z, 179.
— — $\text{ker}_\nu z$, $\text{kei}_\nu z$, 172.
— — $\text{ker}'z$, $\text{kei}'z$, 180.
— — $\text{ker}'_\nu z$, $\text{kei}'_\nu z$, 172.
— — $M_0(z)$, $\theta_0(z)$, 133, 154.
— — $M_1(z)$, $\theta_1(z)$, 133, 154.
— — $N_0(z)$, $\phi_0(z)$, 133, 155.
— — $N_1(z)$, $\phi_1(z)$, 133, 155.
— — $Y_0(z)$, 85, 160, 174.
— — $Y_\nu(z)$, 69, 161.
— — practical application of, 15, 75, 79, 87, 111–13, 131.
— — smallest term in, 70.

ber and bei functions, 119–33.
— — application, 134.
— — asymptotic series, 169.
— — graphs, 120.
— — polar form, 122.
— — tabular values, 177, 178, 181.
$\text{ber}_\nu^2 z + \text{bei}_\nu^2 z$, series for, 153.
Bernoulli's equation for chain, 1, 16.
Bessel coefficients, 1–3, 41.
— — generating function of, 41.
Bessel's definition of $J_n(z)$, 3.
— equation for $J_n(z)$, 3.
Bessel functions of first kind, 4.
— — — — modified, 102, 103.
— — of second kind, 4.
— — — — modified, 102, 103.
— — of third kind, 8.
— — order zero, 5, 6.
— — — n, 20, 102.
— — — $(n+\frac{1}{2})$, 64.
— — — ν, 60, 102, 103.
Bessel loud-speaker horns, 73, 87, 88.
— — — relative performance, 77.
boundary conditions, annular membrane, 14.
— — circular membrane, 11, 23.
— — driven membrane, 13.

cable, electrical data, 112.
capacity of submarine cable, 106, 112.
charge in furnace crucible, 145.
circular functions, comparison with B.F., 3.
— — relationship to B.F., 43–5, 64.
— — series for in terms of B.F., 42.
complete solution of Bessel's equation, 7, 23.
complex numbers, graphical representation of $i^{\frac{1}{2}}$, $i^{\frac{3}{2}}$, etc., 119.
complex variable, 61, 69–73, 82, 85–7, 110–13, 155.
— — altering angle range, 61–3, 72, 86, 111.
— — numerical evaluation of functions, 72, 73, 112.
condenser loud speaker, 12, 18, 19.
— microphone, 17.
conical bar, lateral vibrations, 114.
— — vibrational amplitude, 114.
conical shell, longitudinal vibrations, 118.
current density in straight wire, 134–7.
cylinder functions, definition of, 28.
— — integrals for product of, 96, 97.
— — recurrence formulae, 28, 35, 162.

determinant, Wronskian, 6, 115, 116, 156.
differential equation, Bessel's, 3, 39, 74.
— — Bessel type, 15, 16, 37, 113, 114, 116, 119, 128, 150, 151, 153, 155.
— — for ber z, bei z, 119.
— — for ker z, kei z, 119.

differential equation for $I_0(z)$, $K_0(z)$, 113.
—— for $I_\nu(z)$, $K_\nu(z)$, 102.
—— for $J_0(z)$, $Y_0(z)$, 4.
—— for $J_n(z)$, $Y_n(z)$, 3.
—— for $J_\nu(z)$, $Y_\nu(z)$, 22.
—— Legendre's, 31.
—— for loud-speaker horn, 73.
—— for $M_0(z)$, $N_0(z)$, 128.
—— for $M_1(z)$, $N_1(z)$, 128.
—— method of solution, 5, 6, 30.
—— substitutions to obtain generic form, 15, 16, 26, 32, 37–9, 75, 108, 113–15, 150, 151, 153.
duplex telegraphy, with tapered cable, 106.

eddy current furnace, 147–9.
—— loss in solenoid core, 141–5.
—— —— in straight wire, 140.
effective inductance of cable, 106.
—— —— of straight wire, 140.
—— —— of toroid with core, 145.
effective permeability, 146.
effective resistance of cable, 106.
—— —— of solenoid with core, 145.
—— —— of straight wire, 140.
—— —— of toroid with core, 145.
electrical network, 20, 111.
—— transmission lines, 106–13.
—— —— schematic diagram of, 107.
elliptic planetary motion, 2.
equation, Bessel's, 3.
—— Legendre's, 31.
—— zonal surface harmonics, 29.
Euler's constant (γ), 7, 57.
expansion curve of loud-speaker horn, 73, 74.
expansions in terms of B.F., 42, 43, 50, 51, 160.

factorial function, 58.
flaring of horn, effect of, 78, 79.
—— index of horn, 74.
fluid pressure on disk, 84, 91, 93.
—— —— on sphere, 40.
functions of various kinds, see under B.
fundamental systems of solutions, 6, 20–2, 103.
furnaces, application of B.F. to, 142–9.
— eddy current, 147–9

gamma function $\Gamma(z)$, 57–9, 81, 168.
—— calculation of, 59, 60, 81.
—— graph of, 59.
generating function of Bessel coefficients, 41, 53.

graphical representation of ber z, bei z, 120.
—— $\mathbf{H}_0(z)$, $\mathbf{H}_1(z)$, 66.
—— of $I_0(z)$, $K_0(z)$, 104.
—— of $i^{\frac{1}{2}}$, $i^{\frac{3}{2}}$, etc., 119.
—— of $J_0(z)$, $Y_0(z)$, 8.
—— of $J_n(z)$, 21.
—— of $M_0(z)$, $\theta_0(z)$, 123.
—— of $\mathbf{W}(ma)$, $\Pi(ma)$, 145.

Hankelian functions, 8.
Heaviside-Bessel transmission line, 110, 117.
historical introduction, 1–3.
horns, loud-speaker, 73–80, 87, 88.
hyperbolic functions, 109.
—— relationship to B.F., 64.
hypergeometric function, 56, 57, 80, 81, 91, 92, 168.
$H_0^{(1)}(z)$, $H_0^{(2)}(z)$, definition, 8.
$H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, definition, 23.
—— as solutions of Bessel's equation, 23.
—— asymptotic expansions of, 71, 79, 85, 86, 161, 162.
—— application of to horns, 75, 79.
$\mathbf{H}_0(z)$, asymptotic formula, 176.
— definition, 66.
— graph, 66.
— tabular values, 176.
— zeros, 66.
$\mathbf{H}_1(z)$, asymptotic formula, 176.
— definition, 67, 68.
— graph, 66.
— tabular values, 176.
$\mathbf{H}_\nu(z)$, definition, 67.
— asymptotic expansion, 71, 166.
— comparison with $J_\nu(z)$, 67.
— equation for, 84, 167.
— integral representation, 67, 166.
— series, 67.

imaginary arguments, 102, 103, 106.
infinite integrals, exponential, 18, 55, 160.
—— products of two B.F.s, 91–3, 160.
inductance of cable, 108.
inertia component of sound pressure on disk, 84.
—— —— in horn, 79.
integral representation of $\mathbf{H}_\nu(z)$, 67, 166.
—— of $I_\nu(z)$, 103, 162–4.
—— of $J_\nu(z)$, 65, 157–9.
—— of $K_0(z)$, 116.
—— of $K_\nu(z)$, 165.
—— of $Y_\nu(z)$, 161.

integrals of products of B.F. (Lommel), 94–100, 115, 126, 131, 154, 160, 162, 164, 166.
integration, 43–50.
$I_0(z)$, application, 117, 118.
— definition, 162.
— equation for, 157.
— integral representation, 162.
$I_n(z)$, integral representation, 151, 164.
$I_\nu(z)$, asymptotic expansion, 163.
— definition, 102.
— equation for, 102, 157.
— relationship with $J_\nu(z)$, 103.
$I_{\pm\frac{1}{2}}(z)$, relationship with hyperbolic functions, 115, 164.
Isometric plotting of $J_n(z)$, 21.

$J_0(z)$, application, 11–15, 17, 18, 88.
— asymptotic expansions, 69, 157, 173.
— equation for, 4, 157.
— integral representation, 16, 43–5, 152, 157.
— series for, 5, 6, 157.
— tabular values, 173.
— zeros, 35.
$J_1(z)$, application, 88.
— asymptotic formula, 173.
— tabular values, 173.
— zeros, 35.
$J_2(z)$, $J_3(z)$, $J_4(z)$, tabular values, 174.
$J_n(z)$, application, 88.
— Bessel's definition, 3.
— equation for, 20, 157.
— integral representation, 3, 42, 43, 51, 52, 159.
$J_\nu(z)$, asymptotic expansion, 69, 158.
— definition, 60.
— equation for, 22, 157.
— integral representation, 64, 158, 159.
— series for, 60, 158.
$J_{\frac{1}{2}}(z)$, application of, 109.
— relationship to circular functions, 64.
— hyperbolic functions, 64.
— series for, 64.
$J_{n+\frac{1}{2}}(z)$, relationship to circular functions, 64, 159.
— series for, 159.
— as solution to equation of zonal harmonics, 32.
$J_{\pm\frac{1}{3}}(z)$, $J_{\pm\frac{2}{3}}(z)$, application of, 110–13.
$J_{\frac{1}{3}}(z)$, series for, 83.

ker and kei functions, 119–33.
— — application, 149.
— — asymptotic series, 172, 179, 180.
— — polar form of, 122.
ker and kei, tabular values, 179–81.
— — values of $z < 1$, 179, 180.
$K_0(z)$, application, 117–18.
— asymptotic formula, 164.
— equation for, 157.
— series for, 164.
$K_n(z)$, application, 117–18.
— series for, 165.
$K_\nu(z)$, asymptotic series, 165.
— definition, 103, 165.
— integral representation, 165.
— in terms of $H_\nu^{(1)}(zi)$, 103.

Legendre's coefficients, 31.
— equation, 31.
— functions $P_n(\mu)$, $Q_n(\mu)$, 17, 31.
— graphical representation, 29.
— polynomials, 31.
— recurrence formula, 31.
— table of, 31.
linearly tapered transmission line, 110–13.
linearly independent solutions of Bessel's equation, 6, 103, 156.
loaded submarine cable, 106, 107, 110–12.
Lommel integrals for product of two B.F.s, 94–100, 115, 126, 131, 154, 160, 162, 164, 166.
loss function W(ma), 144.
loud-speaker horns, 73–80, 87, 88.

maximum eddy current loss in core, 144.
mechanical reactance of horn throat, 79.
— resistance of horn throat, 77–9.
membrane, annular, 14, 17, 55.
— circular, 9.
— driven, 12.
— effective mass, 13, 18.
— natural vibrations, 12.
— nodal circles, 12, 14, 27.
— radial nodes, 27.
— shape during vibration, 12, 17, 18, 27.
microphone, condenser, 17.
modulation products, 93, 156 (ex. 70).
$M_0(z)$, $\theta_0(z)$, $M_1(z)$, $\theta_1(z)$, application, 135–41, 143, 146, 148.
— asymptotic formulae, 133.
— graphs, 123.
— tabular values, 182, 183.
$M_\nu(z)$, $\theta_\nu(z)$, 122–31.

Neumann's second solution of Bessel's equation, 6, 7, 23.
nodal circles on membrane, 12, 14, 27.
— — on sphere, 31.
nodes, radial on membrane, 27.

$N_0(z)$, $\phi_0(z)$, $N_1(z)$, $\phi_1(z)$, asymptotic formulae, 133.
$N_\nu(z)$, $\phi_\nu(z)$, 122–31.

penetration function $\Pi(ma)$, 146, 149.
permeability of furnace core, 146–9.
phase of current in conductor, 136.
'phase z', definition, xi.
plane wave assumption in horn theory, xi.
polar curves of sound distribution annulus, 18.
— — of disk, 49, 50, 53, 54.
power delivered to horn, 88.
— factor in horn, 77.
— loss due to eddy currents in core, 142–4.
— — — — straight wire, 140, 141.
products of two B.F.s, integrals, 91, 94–7, 99, 100, 115, 126, 131, 154, 160, 162, 164, 166.
— — — series for, 97, 98.
propagation coefficient of transmission line, 106.

radial modes of membrane, 26, 27.
reactance of horn throat, 79.
rectifying valve, current, 105.
recurrence formulae, cylinder functions, 28, 35, 162.
— — for $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, 26, 161.
— — for $\mathbf{H}_\nu(z)$, 68, 167.
— — for $I_n(z)$, $I_\nu(z)$, 163.
— — for $J_n(z)$, $J_\nu(z)$, 24, 25, 35, 156, 158.
— — for $K_n(z)$, $K_\nu(z)$, 165.
— — for $P_n(\mu)$, 31.
— — for $Y_n(z)$, $Y_\nu(z)$, 26, 161.
resistance, effective, see under E.
— at horn throat, 79.
roots of $\mathbf{H}_0(z) = 0$; 66.
— of $J_0(z) = 0$, $Y_0(z) = 0$; 8, 9.
— of $J_1(z) = 0$; 35.

series impedance of cable, 108–12, 117, 118.
shunt admittance, 108–12, 117, 118.
skin effect in wire, 134–41.
— — in tube, 141.
smallest term in asymptotic expansion, 70.
solutions of Bessel's equation, 4, 6, 20, 21, 157.
Sonine's definition of cylinder function, 28.
— first finite integral, 90.
sound distribution from annulus, 18.
sound distribution from disk, 49, 50, 53, 54, 100.
— power from disk, 99.
spherical harmonics, 28–34, 40, 151.
sphere, vibrating, 29, 40.
squares and products of B.F., series for, 97, 98.
submarine cable, 21, 55, 87, 106–113, 114, 117, 118.
symbols, x.

tabular values of ber z, bei z, 177.
— — of ber$'z$, bei$'z$, 178.
— — of ber$_n z$, bei$_n z$, 181.
— — of ber$'_n z$, bei$'_n z$, 181.
— — of $\mathbf{H}_0(z)$, $\mathbf{H}_1(z)$, 176.
— — of $J_0(z)$, $J_1(z)$, 173.
— — of $J_2(z)$, $J_3(z)$, $J_4(z)$, 175.
— — of ker z, kei z, 179.
— — of ker$'z$, kei$'z$, 180.
— — of ker$_n z$, kei$_n z$, 181.
— — of ker$'_n z$, kei$'_n z$, 181.
— — of $M_0(z)$, $\theta_0(z)$, 182.
— — of $M_1(z)$, $\theta_1(z)$, 183.
— — of $Y_0(z)$, $Y_1(z)$, 174.

uniform transmission line, 109.

vector diagram for $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, 25.
velocity potential, disk, 93.
— — graphical representation, 74, 88.
— — loud-speaker horn, 73–6, 87, 88.
— — sphere, 29, 33–5, 40.
vibrational frequencies, annular membrane, 14, 15.
— — circular membrane, 9–12, 17.
— — driven membrane, 12–14, 18.

wattless component of sound pressure, 25.
— — — — in horn, 79.
Weber-Schafheitlin, infinite integral, 91–3.
Weber's solution of Bessel's equation, 7, 20, 23.
Wronskian determinant, 6, 115, 116, 156.
$\mathbf{W}(ma)$, loss function, 144.

$Y_0(z)$, application, 14, 15, 17, 88.
— asymptotic formula, 85, 160, 174.
— equation for, 4, 157.
— graphical representation, 8, 9.
— series for, 7, 160.
— tabular values, 174.
— zeros, 9.

$Y_1(z)$, application, 88.
— asymptotic formula, 174.
— tabular values, 174.
$Y_n(z)$, series for, 7, 161.
$Y_\nu(z)$, asymptotic series, 70, 161.
— definition, 22, 161.
— equation for, 22.
— integral representation, 161.
$Y_{\frac{1}{2}}(z)$, relation to circular functions, 64, 161.

zeros of various functions, see under 'roots'.
zonal surface harmonics, 28–34.
— — — application, 40.
— — — graphical representation, 29.